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EFFECT OF ASPECT RATIO ON THE AIR FORCES AND MOMENTS OF HARMONICALLY OSCILLATING THIN RECTANGULAR WINGS IN SUPERSONIC POTENTIAL FLOW

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SUMMARY

This paper treats the effect of aspect ratio on the air forces and moments of an oscillating flat rectangular wing in supersonic potential flow. The linearized velocity potential for the wing undergoing sinusoidal torsional oscillations simultaneously with sinusoidal vertical translations is derived in the form of a power series in terms of a frequency parameter. The series development is such that the differential equation for the velocity potential is satisfied to the required power of the frequency parameter considered and the linear boundary conditions are satisfied exactly. The method of solution can be utilized for other plan forms, that is plan forms for which certain steady-state solutions are known.

Simple, closed expressions that include the reduced frequency to the third power, which is thought to be sufficient for most practical applications, are given for the velocity potential, the components of total force and moment coefficients, and the components of chordwise section force and moment coefficients. The components of total force and moment coefficients indicate the over-all effect of aspect ratio on these quantities; however, the components of chordwise coefficients yield more information because they account for the spanwise distribution of aerodynamic loading of a rectangular wing and may therefore be more useful for flutter calculations. It is found that the components of force and moment coefficients for a small-aspect-ratio wing may deviate considerably from those for an infinite-aspect-ratio wing. Thickness effects which may alter some of the conclusions are not taken into account in the analysis. Results of some selected calculations are presented in several figures and discussed.

INTRODUCTION

The effect of aspect ratio on the single-degree torsional instability of a finite rectangular wing oscillating in a supersonic stream was treated in reference 1 by expanding, in powers of the frequency of oscillation, the linearized velocity potential developed in reference 2. Since only slow oscillations were considered pertinent to single-degree torsional instability, terms in the expansion involving the frequency of oscillation to powers higher than the first were not considered.

In the present paper the expanded linearized velocity potential is used to study the effect of aspect ratio on the air forces and moments of an oscillating, thin, flat, finite, rectangular wing when higher powers of the frequency of oscillation are taken into account. The motions considered are sinusoidal torsional oscillations about a spanwise axis taken simultaneously with sinusoidal vertical translations of this axis. The velocity potential is developed by superpositions of sources and doublets, so as to include all powers of the frequency of oscillations up to any desired power. Simple, closed expressions are given for the velocity potential, components of the total force and moment coefficients, and components of the chordwise section force and moment coefficients involving powers of the frequency up to and including the third power. Extension of the results to include higher powers of the frequency is straightforward.

A recent publication, reference 3, that became available after this investigation was completed, is partly devoted to the treatment of a rectangular wing undergoing the same types of harmonic motions as those considered herein. The velocity potential is determined in the form of a double integral, by application of the Fourier transform to the boundary-value problem for this potential, and expressions for forces and moments are given in terms of this double integral. The reduction of the integral expressions of reference 3 to forms desirable for flutter calculations, that is, chordwise section forces and moments, is not given.

SYMBOLS

ϕ disturbance-velocity potential

x,y,z rectangular coordinates attached to wing moving in negative x-direction

ξ,η	rectangular coordinates used to represent space location of sources or doublets in xy-plane
Z_m	function defining mean ordinates of any chordwise section of wing such as $y = y_1$ as shown in figure 1
$w(x,y_1,t)$	vertical velocity at surface of wing along chordwise section at $y = y_1$
x_0	abscissa of axis of rotation of wing (elastic axis) as shown in figure 1
t	time
h	vertical displacement of axis of rotation
h_0	amplitude of vertical displacement of axis of rotation, positive downward
α	angle of attack
α_0	amplitude of angular displacement about axis of rotation, positive leading edge up
$\dot{h},\dot{\alpha}$	time derivative of h and α, respectively
V	velocity of main stream
c	velocity of sound
M	free-stream Mach number (V/c)
$\beta = \sqrt{M^2 - 1}$	
$\tau_1,\tau_2,\eta_1,\eta_2$	functions defined with equation (7)
$W(\xi,\eta)$	function used to represent space variation of source and doublet strengths
$w(t)$	function used to represent time variation of source and doublet strengths
ω	frequency of oscillation
$\overline{\omega} = \dfrac{M^2\omega}{V\beta^2}$	
k	reduced frequency $(\omega b/V)$

$R = \beta\sqrt{(\eta - \eta_1)(\eta_2 - \eta)}$

a_{nm} represents functions of $\overline{\omega}$, x, and M, defined in equation (15)

f_1,f_2,f_3 represent functions of x, x_0, and $\overline{\omega}$, defined in equation (19)

D_i function used to denote doublet distributions (see equation (21))

F_n function defined in equation (28)

G_n function defined in equation (29)

ρ density

Δp local pressure difference measured positive downward, defined in equation (31)

b half-chord

s half-span

A aspect ratio (s/b)

$\overline{P}$ total force acting on wing defined in equation (32)

$\overline{L_1},\overline{L_2},\overline{L_3},\overline{L_4}$ components of total force coefficients, defined in equation (35)

$\overline{M_\alpha}$ total moment acting on wing, defined in equation (36)

$\overline{M_1},\overline{M_2},\overline{M_3},\overline{M_4}$ components of total moment coefficients, defined in equation (38)

P section force (total force at any spanwise station), defined in equation (39)

L_1,L_2,L_3,L_4 components of section force coefficients, defined in equations (41) and (42)

M_α section moment (total moment at any spanwise station), defined in equation (40)

M_1,M_2,M_3,M_4 components of section moment coefficients, defined in equations (43) and (44)

$\overline{\overline{F}}_n,\overline{\overline{G}}_n$ functions related to F_n and G_n, defined in appendix

ANALYSIS

Boundary-Value Problems for Velocity Potentials

Consider a thin flat rectangular wing moving at a constant supersonic speed in a chordwise direction normal to its leading edge as shown in figure 1. The boundary-value problems for the velocity potential for such a wing may be conveniently classified into two types associated with the nature of the flow over different portions of the wing. On the portion of the wing between the Mach cones emanating from the foremost point of each tip (region N in fig. 1(a)) there is no interaction between the flow on the upper and lower surfaces of the wing. The type of boundary-value problem for this portion of the wing is referred to herein as "purely supersonic" and the velocity potential for region N is denoted by ϕ_N. On portions of the wing within the Mach cones emanating from the foremost point of each tip (regions T_1, T_2, and T_3 in fig. 1(a)), there is interaction between the flow on the upper and lower surfaces of the wing. The type of boundary-value problem for these portions of the wing is referred to as "mixed supersonic" and the velocity potentials for these regions are designated by ϕ_{T1}, ϕ_{T2}, and ϕ_{T3}, respectively. The complete velocity potential at a point may then be expressed as ϕ_N, ϕ_{T_1}, ϕ_{T_2}, or ϕ_{T_3} according to the region that contains the point.

As customary in linear theory, as applied to thin flat surfaces, the boundary conditions are to be ultimately satisfied by the velocity potentials at the projection of the wing onto a plane (the xy-plane) with respect to which all deflections are considered small and which lies parallel to the free-stream direction. Thickness effects are not taken into account; hence, the velocity potentials are associated only with conditions that yield lift and are consequently antisymmetrical with respect to the plane of the projected wing. It is therefore necessary to consider the potentials at only one surface, upper or lower, of the projected wing. The upper surface is chosen for this analysis.

The differential equation for the propagation of small disturbances that must be satisfied by the velocity potentials is (when referred to a rectangular coordinate system x,y,z with the xy-plane coincident with the reference plane and moving uniformly in the negative x-direction, fig. 1)

$$\frac{1}{c^2}\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)^2 \phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \tag{1}$$

The boundary conditions that must be satisfied by the velocity potential are: (a) In regions T_1, T_2, T_3, and N the flow must be tangent to the surface of the wing or

$$\left(\frac{\partial \phi}{\partial z}\right)_{z \longrightarrow 0} = w(x,y_1,t) = V \frac{\partial Z_m}{\partial x} + \frac{\partial Z_m}{\partial t} \tag{2}$$

where Z_m is the vertical displacement of the ordinates of the surface of any chordwise section of the wing (see fig. 1(b)). (b) In regions T_1 and T_2 the pressure must fall to zero along the wing tips and remain zero in the portion of the Mach cones emanating from the foremost points of the wing tips not occupied by the wing. (Another condition, that the potential must be zero ahead of the wing and in the regions off the wing adjacent to the Mach cones emanating from the foremost points of the tips, is automatically satisfied by the type of source and doublet synthesis employed in the solutions.)

For the particular case of a wing independently performing small sinusoidal torsional oscillations of amplitude $|\alpha_0|$ and frequency ω about some spanwise axis x_0 and small sinusoidal vertical translations of amplitude $|h_0|$ and frequency ω, the equation of Z_m is

$$Z_m = e^{i\omega t}\left[\alpha_0(x - x_0) + h_0\right] = \alpha(x - x_0) + h \tag{3}$$

Substituting this expression for Z_m into equation (2) gives

$$w(x,y_1,t) = V\alpha + \dot{\alpha}(x - x_0) + \dot{h} \tag{4}$$

The velocity potential may thus be expressed as the sum of separate effects due to position and motion of the wing associated with the individual terms in equation (4) as

$$\phi = \phi_\alpha + \phi_{\dot{\alpha}} + \phi_{\dot{h}} \tag{5}$$

Derivation of ϕ_N

The boundary-value problem in the purely supersonic region (fig. 2(a)) is the same as that for the two-dimensional wing treated in reference 4. This problem is there shown to be satisfied by a distribution of sources referred to, in this case, as moving sources because of the uniform motion; that is,

$$\phi_N(x,y,z,t) = -\frac{1}{2\beta\pi}\int_0^{x-\beta z}\int_{\eta_1}^{\eta_2} W(\xi,\eta)\phi_1 \, d\eta \, d\xi \tag{6}$$

In equation (6) $W(\xi,\eta)$ represents the space variation of source strength and must be evaluated in accordance with the individual terms of equation (4) and ϕ_1 is the potential of a moving source situated at the point $(\xi,\eta,0)$ that may be expressed as

$$\phi_1 = \frac{w(t - \tau_1) + w(t - \tau_2)}{\sqrt{(\eta - \eta_1)(\eta_2 - \eta)}} \tag{7}$$

where $w(t)$ is the time variation of source strength and the symbols with subscripts appearing in equation (7) are defined as

$$\tau_1 = \frac{M(x - \xi)}{c\beta^2} - \frac{\sqrt{(\eta - \eta_1)(\eta_2 - \eta)}}{\beta c}$$

$$\tau_2 = \frac{M(x - \xi)}{c\beta^2} + \frac{\sqrt{(\eta - \eta_1)(\eta_2 - \eta)}}{\beta c}$$

$$\eta_1 = y - \frac{1}{\beta}\sqrt{(x - \xi)^2 - \beta^2 z^2}$$

$$\eta_2 = y + \frac{1}{\beta}\sqrt{(x - \xi)^2 - \beta^2 z^2}$$

The time variation of source strength $w(t)$ for harmonic oscillations may be written as

$$w(t) = e^{i\omega t} \tag{8}$$

The numerator in equation (7) thus becomes

$$\begin{aligned} w(t - \tau_1) + w(t - \tau_2) &= e^{i\omega(t-\tau_1)} + e^{i\omega(t-\tau_2)} \\ &= 2e^{i\omega t} \, e^{-i\omega\frac{\tau_2+\tau_1}{2}} \cos \omega \frac{\tau_2 - \tau_1}{2} \end{aligned} \tag{9}$$

Substituting equations (7) and (9) into equation (6) yields

$$\phi_N(x,y,z,t) = -\frac{e^{i\omega t}}{\pi}\int_0^{x-\beta z}\int_{\eta_1}^{\eta_2}\frac{W(\xi,\eta)e^{-i\bar{\omega}(x-\xi)}\cos\left(\frac{\bar{\omega}}{M}R\right)d\eta\,d\xi}{R} \tag{10}$$

where, for briefness,

$$\bar{\omega} = \frac{\omega M}{c\beta^2} = \frac{M^2\omega}{V\beta^2}$$

and

$$R = \sqrt{(x-\xi)^2 - \beta^2(y-\eta)^2 - \beta^2 z^2} = \beta\sqrt{(\eta-\eta_1)(\eta_2-\eta)}$$

The values of $W(\xi,\eta)$ associated with the different terms of equation (4) are, in the order in which they are used:

For $\dot{h}$

$$W(\xi,\eta) = \frac{iV\beta^2\bar{\omega}}{M^2}h_0 \tag{11}$$

For $V\alpha$

$$W(\xi,\eta) = V\alpha_0 \tag{12}$$

For $\dot{\alpha}(x - x_0)$

$$W(\xi,\eta) = \frac{iV\beta^2\bar{\omega}}{M^2}\alpha_0(\xi - x_0) \tag{13}$$

If any of the values of $W(\xi,\eta)$ given in equations (11), (12), and (13) is put into equation (10), the integration with respect to η can be readily performed and the remaining integral evaluated as a series of Bessel functions. (See, for example, reference 4.) However, in order to be consistent with and to lead naturally to a succeeding part of the analysis the integrand is expanded into a Maclaurin's series with respect to $\bar{\omega}$. The expansion yields

$$\phi_N(x,y,z,t) = -\frac{e^{i\omega t}}{\pi}\int_0^{x-\beta z}\int_{\eta_1}^{\eta_2} W(\xi,\eta)\left[\left(a_{01}\frac{1}{R} + a_{11}\frac{\xi}{R} + \ldots +\right.\right.$$

$$\left. a_{n1}\frac{\xi^n}{R} + \ldots\right) + \left(a_{02}R + a_{12}\xi R + \ldots + a_{n2}\xi^n R + \ldots\right) + \ldots +$$

$$\left.\left(a_{0m}R^{2m-3} + a_{1m}\xi R^{2m-3} + \ldots + a_{nm}\xi^n R^{2m-3} + \ldots\right) + \ldots\right] d\eta\, d\xi \quad (14)$$

where the coefficients a_{nm} are functions of $\overline{\omega}$, x, and M; those coefficients involving $\overline{\omega}$, up to and including the third power, are:

$$\left.\begin{aligned}
a_{01} &= 1 - i\overline{\omega}x - \frac{\overline{\omega}^2}{2}x^2 + \frac{i\overline{\omega}^3}{6}x^3 \\
a_{11} &= i\overline{\omega} + \overline{\omega}^2 x - \frac{i\overline{\omega}^3}{2}x^2 \\
a_{21} &= -\frac{\overline{\omega}^2}{2} + \frac{i\overline{\omega}^3}{2}x \\
a_{31} &= -\frac{i\overline{\omega}^3}{6} \\
a_{02} &= -\frac{\overline{\omega}^2}{2M^2} + \frac{i\overline{\omega}^3}{2M^2}x \\
a_{12} &= -\frac{i\overline{\omega}^3}{2M^2}
\end{aligned}\right\} \quad (15)$$

Observe the following identity that is valid regardless of the highest power of $\overline{\omega}$ considered and that will be of subsequent use, namely

$$a_{01} + xa_{11} + \ldots + x^n a_{n1} \equiv 1 \quad (16)$$

It will be noted in equation (14) that the potential of a moving source when expanded in terms of the frequency appears as a series of terms similar to steady-state source potentials plus series of terms involving various powers of R. By grouping the terms in equation (14) with respect to powers of ξ, the following form of the source potential convenient for later use is obtained:

$$\phi_N(x,y,z,t) = -\frac{e^{i\omega t}}{\pi}\int_0^{x-\beta z}\int_{\eta_1}^{\eta_2} W(\xi,\eta)\left[\left(a_{01}\frac{1}{R} + a_{02}R + \ldots + a_{0m}R^{2m-3} + \ldots\right) + \xi\left(a_{11}\frac{1}{R} + a_{12}R + \ldots + a_{1m}R^{2m-3} + \ldots\right) + \ldots + \xi^n\left(a_{n1}\frac{1}{R} + a_{n2}R + \ldots + a_{nm}R^{2m-3} + \ldots\right) + \ldots\right]d\eta\, d\xi \tag{17}$$

With the terms of the series grouped in this manner, in view of the fact that the differential equation (1) is independent of ξ, it is apparent that the coefficient of each power of ξ in equation (17) is a solution to the differential equation.

If the values of $W(\xi,\eta)$ in equations (11), (12), and (13) are put into either equation (14) or equation (17), the integrations of each term can be easily carried out in closed form. Moreover it can readily be shown that, when all the terms involving $\bar{\omega}$ up to a given power are taken into account, the differential equation (1) is satisfied to the highest power of $\bar{\omega}$ considered. The boundary condition of tangential flow as expressed in equation (4) is satisfied exactly and does not depend on the order of $\bar{\omega}$ considered.

Putting the values of $W(\xi,\eta)$ in equations (11), (12), and (13) successively into either equation (14) or equation (17), carrying out the indicated integration, and setting $z = 0$ yields for the velocity potential, to the third power of $\bar{\omega}$ at the upper surface of the wing, in the purely supersonic region:

$$\phi_N = -\frac{1}{\beta}\left(\dot{h}f_1 + V\alpha f_2 + \dot{\alpha}f_3\right) \tag{18}$$

where

$$\left.\begin{aligned}
f_1 &= x - \frac{i\bar{\omega}}{2} x^2 - \frac{\bar{\omega}^2}{12M^2} (2\beta^2 + 3)x^3 \\
f_2 &= x - \frac{i\bar{\omega}}{2} x^2 - \frac{\bar{\omega}^2}{12M^2} (2\beta^2 + 3)x^3 + \frac{i\bar{\omega}^3}{48M^2} (2\beta^2 + 5)x^4 \\
f_3 &= \frac{x}{2} (x - 2x_0) - \frac{i\bar{\omega}x^2}{6} (x - 3x_0) - \frac{\bar{\omega}^2}{48M^2} (2\beta^2 + 3)x^3(x - 4x_0)
\end{aligned}\right\} \tag{19}$$

Derivation of ϕ_{T_1}, ϕ_{T_2}, and ϕ_{T_3}

For convenience in the derivation of the velocity potentials in the regions of mixed supersonic flow the coordinate system is chosen with the origin at one tip and the y-axis coincident with the wing leading edge. (See fig. 2(b).) Region T_3 (shown in fig. 1(a)) exists whenever the Mach lines from the tips intersect one another on the wing (that is, when $A\beta < 2$). When the Mach lines from the tip intersect one another on the wing and in addition intersect the tips ahead of the trailing edge (when $A\beta < 1$), other regions, not considered herein, have to be taken into account, and the determination of the velocity potential becomes very cumbersome. This discussion is restricted to the condition that the aspect ratio be greater than or equal to $1/\beta$, that is, $A\beta \geqq 1$.

Because of the similarity of conditions in regions T_1 and T_2, the potentials ϕ_{T_1} and ϕ_{T_2} will be of the same form. It is therefore only necessary to derive one of these, say ϕ_{T_1}. The other potential can then be obtained by a simple translation of variables. The potential ϕ_{T_3}, as given later, is a linear combination of ϕ_N, ϕ_{T_1}, and ϕ_{T_2}.

As pointed out in reference 1, moving doublets may be used to satisfy the boundary-value problem in the regions of mixed supersonic flow. The potential of the type of doublet required may be obtained by partial differentiation of the potential of a moving source with respect to the direction normal to the plane of the wing (in the present notation with respect to z). If the expanded form of the unit source potential appearing in equation (17) is partially differentiated with respect to z, the following expanded form of a unit doublet potential is obtained:

$$\phi_2 = \frac{e^{i\omega t}}{\pi} \frac{\partial}{\partial z}\left[\left(a_{01}\frac{1}{R} + a_{02}R + \ldots + a_{0m}R^{2m-3} + \ldots\right) + \xi\left(a_{11}\frac{1}{R} + a_{12}R + \ldots + a_{1m}R^{2m-3} + \ldots\right) + \ldots + \xi^n\left(a_{n1}\frac{1}{R} + a_{n2}R + \ldots + a_{nm}R^{2m-3} + \ldots\right) + \ldots\right] d\eta\, d\xi \tag{20}$$

Since the coefficient of each power of ξ in equation (20) satisfies the differential equation and since the differential equation is linear, it is permissible to weight these coefficients differently and therefore write the velocity potential for region T_1 as

$$\phi_{T_1} = -\frac{e^{i\omega t}}{\pi}\frac{\partial}{\partial z}\int_r\int W(\xi,\eta)\left[D_0\left(a_{01}\frac{1}{R} + a_{02}R + \ldots + a_{0m}R^{2m-3} + \ldots\right) + D_1\left(a_{11}\frac{1}{R} + a_{12}R + \ldots + a_{1m}R^{2m-3} + \ldots\right)\xi + \ldots + D_n\left(a_{n1}\frac{1}{R} + a_{n2}R + \ldots + a_{nm}R^{2m-3} + \ldots\right)\xi^n + \ldots\right] d\eta\, d\xi \tag{21}$$

where $D_i(\xi,\eta)$ $(i = 0,1,2,\ldots n)$ are the weight factors or distribution functions and where the region of integration r is the portion of the wing lying in the forecone emanating from the point (x,y,z). (See fig. 2(b) for the case $z = 0$.)

The problem is now resolved to that of determining the distribution function for each series in equation (21) so as to make this equation satisfy the boundary conditions. The determination of these functions can be made with the aid of analogy with known steady-state solutions. Examination of equation (21) shows that the first term of each series has the form of a steady-state doublet potential and, as discussed subsequently, is therefore a type of singularity that can be superposed to satisfy certain conditions of tangential flow for the antisymmetrical type of problem in regions of mixed supersonic flow. The second term of each series, with the indicated differentiation performed, is a singular type of potential, (steady-state source type); however, it is multiplied by z and would therefore yield zero vertical velocity at the xy-plane. All other terms appearing in each series in equation (21) are similarly multiplied by z and are nonsingular; they therefore yield zero vertical velocity at the xy-plane. In order to satisfy the condition of tangential flow it is therefore necessary to consider only the first term in each series.

Retaining, as necessary, only the first term in each series in equation (21) and imposing the condition of tangential flow as expressed in equation (2) results in an integral equation for the distribution functions D_i $(i = 0,1,2,\ldots n)$ as follows:

$$w(x,y,z,t) = \left(\frac{\partial \phi_{T_1}}{\partial z}\right)_{z \to 0}$$

$$= \lim_{z \to 0} \frac{e^{i\omega t}}{\pi} \frac{\partial^2}{\partial z^2} \int_{\tau}\int W(\xi,\eta) \left(D_0 a_{01} \frac{1}{R} + D_1 a_{11} \frac{\xi}{R} + \ldots + D_n a_{n1} \frac{\xi^n}{R} + \ldots\right) d\eta \, d\xi \tag{22}$$

Consider the following integral which is of the type appearing in a representative "first" term in equation (21) and which represents, as previously mentioned, the potential of a doublet distribution for the steady-flow problem:

$$\frac{\partial}{\partial z} \int\int D_K \frac{\xi^K}{R} \, d\xi \, d\eta \qquad (K \geq 0) \tag{23}$$

In steady flow a distribution of this type is convenient for treating the problem of satisfying the condition of tangential flow for a distribution of normal velocity at the wing surface prescribed independent of y but proportional to x^K. The required integral equation for the distribution function D_K in the steady case is

$$\lim_{z \to \infty} \frac{\partial^2}{\partial z^2} \int\int D_K \frac{\xi^K}{R} \, d\xi \, d\eta = Cx^K \tag{24}$$

where C is a known constant.

As noted in the following, the distribution function D_K may be known provided D_0 is known. The value of D_0 in equation (24) is known for a variety of wing plan forms. In the case of rectangular wings the expression for D_0 is derived in reference 1 and found to be

$$D_0 = \frac{2}{\beta}\left[\sqrt{\beta\eta(\xi - \beta\eta)} + \xi \sin^{-1}\sqrt{\frac{\beta\eta}{\xi}}\right] \tag{25}$$

This expression may be considered as a key for finding any other distribution function D_K $(K > 0)$ required to satisfy equation (24) for the rectangular wing. It follows by direct substitution and reduction that any D_K $(K > 0)$ for this wing may be written in terms of D_0, equation (25), as

$$D_K = K!\int_0^{\xi}\int_0^{\xi} \cdots \int_0^{\xi} D_0(d\xi)^K \tag{26}$$

The boundary conditions (a) and (b) can be shown to be satisfied by equation (22) after the values of D_K, given in equation (26), and the values of $W(\xi,\eta)$, defined by equations (11), (12), and (13), are substituted into equation (21) for the velocity potential and the identity in equation (16) is utilized. The velocity potential ϕ_{T_1} is thus determined by equation (21) and these substitutions. At the upper surface of the wing the velocity potential ϕ_{T_1} is given by the expression

$$\begin{aligned}
\phi_{T_1} = -\frac{1}{\beta\pi}\Bigg(&\dot{h}\left[2F_1 - \left(2i\bar{\omega} + \frac{\bar{\omega}^2 x}{M^2}\right)F_2 - \frac{\beta^2\bar{\omega}^2}{M^2}F_3\right] + \\
&V\alpha\left[2F_1 - \left(2i\bar{\omega} + \frac{\bar{\omega}^2}{M^2}x - \frac{i\bar{\omega}^3}{2M^2}\right)F_2 - \frac{\beta^2\bar{\omega}^2}{M^2}F_3 + \frac{i\bar{\omega}^3}{6M^2}(2\beta^2 - 1)F_4\right] + \\
&\dot{\alpha}\Bigg\{2(x - x_0)F_1 - \left[2 + 2i\bar{\omega}(x - x_0) + \frac{\bar{\omega}^2}{2M^2}(x^2 - 2xx_0)\right]F_2 + \\
&\left[2i\bar{\omega} - \frac{\beta^2\bar{\omega}^2}{M^2}(x - x_0)\right]F_3 + \frac{\bar{\omega}^2}{2M^2}(2\beta^2 + 1)F_4\Bigg\}\Bigg)
\end{aligned} \tag{27}$$

where the terms are grouped conveniently by the definition of F_n in the following integral:

$$F_n = \int_0^x x^{n-1}\sin^{-1}\sqrt{\frac{\beta y}{x}}\,dx \qquad (n = 1,2,3,4) \tag{28}$$

(The functions F_n, given in equation (28), and certain related functions are of particular importance in the remainder of this development. Integrated values of this function for the first few values of n and expressions for related functions needed later in this analysis are given in the appendix.)

Examination of equation (27) shows that along the Mach line $x = \beta y$, separating region T_1 from region N, the expression ϕ_{T_1} reduces to the expression for ϕ_N given in equation (18).

The corresponding potentials for regions T_2 and T_3 can now be obtained. The potential ϕ_{T_2} is obtained from equation (27) by merely substituting $2s - y$ for y in equation (28) so that

$$G_n = \int_0^x x^{n-1} \sin^{-1}\sqrt{\frac{\beta(2s - y)}{x}}\, dx \qquad (n = 1, 2, 3, 4) \tag{29}$$

The potential in region T_3 (that is for $1 \leqq A\beta < 2$) is a simple superposition of the potentials for regions N, T_1, and T_2, as in the steady case (see, for example, reference 5), and may be written as

$$\phi_{T_3} = \phi_{T_1} + \phi_{T_2} - \phi_N \tag{30}$$

Forces and Moments

Two types of force and moment coefficients will be derived. First, in order to gain some insight into the over-all effect of aspect ratio on the forces and moments, expressions for total force and moment coefficients are derived. Then, in order to present expressions that are more suitable for use in flutter calculations, expressions for section force and moment coefficients for any station along the span will be derived.

Total forces and moments.- The local pressure difference between the upper and lower surfaces on the wing may be written

$$\Delta p = -2\rho\left(\frac{\partial\phi}{\partial t} + V\frac{\partial\phi}{\partial x}\right) \tag{31}$$

In order to derive expressions for total forces and total moments it is only necessary to consider the velocity potential in two regions; either regions N and T_1 or regions N and T_2. Therefore the expression for the total force, positive downward, on the wing may be written as

$$\overline{P} = -2 \int_N \int \Delta p_N \, dy \, dx - 2 \int_{T_1} \int \Delta p_{T_1} \, dy \, dx \tag{32}$$

where Δp_N is to be calculated from equation (18) and the integrations in the first term are to be extended over the shaded portion of region N shown in figure 2(a), and where Δp_{T_1} is to be calculated from equation (27) and the integrations in the second term are to be extended over region T_1. (The integrations in the first term are simple and may be performed by inspection. Those in the second term may be readily performed by making use of the relations given in the appendix.)

After the indicated integrations have been performed and all position coordinates involved have been referred to the chord 2b (but the original coordinate symbols maintained), the results can be written as

$$\overline{P} = -8\rho b^2 V^2 k^2 A e^{i\omega t} \left[\frac{h_0}{b}\left(\overline{L_1} + i\overline{L_2}\right) + \alpha_0\left(\overline{L_3} + i\overline{L_4}\right)\right] \tag{33}$$

where the reduced frequency k is related to ω and $\overline{\omega}$ by the relations

$$k = \frac{b\omega}{V} = \frac{b\beta^2}{M^2} \overline{\omega} \tag{34}$$

and where

$$\overline{L_1} = \frac{1}{3\beta^4}\left(3\beta - \frac{2 + \beta^2}{A}\right) \tag{35a}$$

$$\overline{L_2} = \frac{1}{\beta k} - \frac{M^2 k}{\beta^5} - \frac{1}{2A}\left[\frac{1}{\beta^2 k} - \frac{M^2 k}{3\beta^6} (4 + \beta^2)\right] \tag{35b}$$

$$\overline{L_3} = \frac{1}{\beta k^2} - \frac{1}{3\beta^5}\left(3 + \beta^2 + 6\beta^2 x_0\right) - \frac{1}{2A}\left\{\frac{1}{\beta^2 k^2} - \frac{1}{3\beta^6}\left[\left(3\beta^2 + 4\right) + 4\beta^2 x_0(2 + \beta^2)\right]\right\} \tag{35c}$$

$$\overline{L_4} = \frac{1}{\beta^3 k}(\beta^2 - 1 - 2\beta^2 x_0) + \frac{M^2 k}{6\beta^7}(5 + \beta^2 + 12\beta^2 x_0) + \frac{1}{3A}\left[\frac{1}{\beta^4 k}(2 + 3\beta^2 x_0) - \frac{M^2 k}{5\beta^8}(8 + 4\beta^2 + 20\beta^2 x_0 + 5\beta^4 x_0)\right] \tag{35d}$$

The quanities $\overline{L_i}$ $(i = 1,2,3,4)$ are the in-phase and out-of-phase components of the total force coefficients, $\overline{L_1}$ and $\overline{L_3}$ being the in-phase and $\overline{L_2}$ and $\overline{L_4}$, the out-of-phase components. It will be noted that $\overline{L_1}$ and $\overline{L_2}$ are associated only with vertical translations of the wing and are independent of axis-of-rotation location x_0. The components $\overline{L_3}$ and $\overline{L_4}$ are associated with angular position and rotation of the wing about any axis $x = x_0$ and depend partly on the location of x_0.

The total moment, positive clockwise, on the wing about the arbitrary axis of rotation $x = x_0$ is

$$\overline{M_\alpha} = -2\int_N\int (x - x_0)\, \Delta p_N\, dy\, dx - 2\int_{T_1}\int (x - x_0)\, \Delta p_{T_1}\, dy\, dx \tag{36}$$

If steps similar to those required to obtain equation (33) are performed, there is obtained

$$\overline{M_\alpha} = -8\rho b^3 V^2 k^2 A e^{i\omega t}\left[\frac{h_0}{b}(\overline{M_1} + i\overline{M_2}) + \alpha_0(\overline{M_3} + i\overline{M_4})\right] \tag{37}$$

where

$$\overline{M_1} = \frac{2}{3\beta^3}(2 - 3x_0) - \frac{1}{6A\beta^4}\left[3\beta^2 + 1 - 4x_0(2 + \beta^2)\right] \tag{38a}$$

$$\overline{M_2} = \frac{1}{\beta k}(1 - 2x_0) - \frac{M^2 k}{2\beta^5}(3 - 4x_0) - \frac{1}{3A}\left\{\frac{1}{\beta^2 k}(2 - 3x_0) - \frac{M^2 k}{5\beta^6}\left[4(\beta^2 + 4)(4 - 5x_0)\right]\right\} \tag{38b}$$

$$\overline{M_3} = \frac{1}{\beta k^2}(1 - 2x_0) - \frac{1}{2\beta^5}\left[3 + \beta^2 + 4x_0(\beta^2 - 1) - 8\beta^2 x_0^2\right] -$$

$$\frac{1}{3A}\left\{\frac{1}{\beta^2 k^2}(2 - 3x_0) - \frac{1}{5\beta^6}\left[4(4 + 3\beta^2) + 5x_0(3\beta^4 + 3\beta^2 - 4) - \right.\right.$$

$$\left.\left.20\beta^2 x_0^2(\beta^2 + 2)\right]\right\} \tag{38c}$$

$$\overline{M_4} = \frac{2}{3\beta^3 k}\left[2(\beta^2 - 1) - 3x_0(2\beta^2 - 1) + 6\beta^2 x_0^2\right] + \frac{M^2 k}{15\beta^7}\left(20 + 4\beta^2 - 25x_0 + \right.$$

$$\left.40\beta^2 x_0 - 60\beta^2 x_0^2\right) + \frac{1}{3A}\left\{\frac{1}{\beta^4 k}\left[3 + 4x_0(\beta^2 - 1) - 6\beta^2 x_0^2\right] - \right.$$

$$\left.\frac{M^2 k}{15\beta^8}\left[20(2 + \beta^2) - 24x_0(2 - 3\beta^2 - \beta^4) - 30\beta^2 x_0^2(4 + \beta^2)\right]\right\} \tag{38d}$$

The quantities $\overline{M_1}$ and $\overline{M_2}$ are, respectively, the in-phase and out-of-phase components of total moment coefficients about the axis $x = x_0$ associated with vertical translations of the wing; $\overline{M_3}$ and $\overline{M_4}$ are the corresponding components due to angular position and rotation of the wing about $x = x_0$.

It is of interest to note in equations (35) and (38) that the components $\overline{L_1}$ and $\overline{M_1}$ do not involve the reduced frequency k. The effect of frequency on these two components comes from terms involving the frequency to the fourth and higher powers; but for values of k thought likely to be encountered in supersonic flutter ($k < 0.1$), the contribution of these higher-powered terms to any of the components in equations (35) and (38) is, for the most part, negligible.

Section forces and moments.- The section forces and moments at any spanwise station are derived by integrating the pressure difference along the chord for the forces, and the pressure difference multiplied by a moment arm for the moments. Since the distribution over the entire wing is symmetrical with regard to the midspan section, it is only necessary to derive expressions for the forces and moments at any station of the half-span adjacent to the origin. (See figs. 1, 2, and 3.)

Under the restrictions previously stipulated, two cases that can arise are considered (see fig. 3): (1) the Mach lines from the tips do not intersect on the wing (or $A\beta > 2$) and (2) the Mach lines intersect on the wing but the Mach line from one tip does not intersect the opposite tip ahead of the trailing edge (or $1 \leqq A\beta \leqq 2$). Only the final forms of the section force and moment equations are given. These forms are easily calculated by deriving the pressure difference for the different regions from the appropriate velocity potential, making use of figure 3 to determine the limits of integration for the regions involved, and using the relations given in the appendix to carry out the more troublesome integrations. The integrated expression for any region can then be reduced to the forms

$$P = -4\rho bV^2k^2e^{i\omega t}\left[\frac{h_0}{b}(L_1 + iL_2) + \alpha_0(L_3 + iL_4)\right] \tag{39}$$

and

$$M_\alpha = -4\rho b^2V^2k^2e^{i\omega t}\left[\frac{h_0}{b}(M_1 + iM_2) + \alpha_0(M_3 + iM_4)\right] \tag{40}$$

where the position coordinates are referred to the chord length $2b$.

The components of force and moment coefficients for the half-span adjacent to the origin are as follows:

Case 1 (see fig. 3(a)): For any section between the tip and the point where the Mach line intersects the trailing edge, or where $0 < y < \frac{1}{\beta}$, the components of section force coefficients are

$$L_1 = -\frac{4}{\pi}\left(\bar{\bar{F}}_1 - \frac{1 + 2\beta^2}{\beta^2}\bar{\bar{F}}_2\right) \tag{41a}$$

$$L_2 = \frac{4}{\beta\pi}\left\{\frac{1}{2k}\bar{\bar{F}}_1 + \frac{M^2k}{\beta^4}\left[(2\beta^2 - 1)\bar{\bar{F}}_2 - 3\beta^2\bar{\bar{F}}_3\right]\right\} \tag{41b}$$

$$L_3 = \frac{4}{\beta\pi}\left[\frac{1}{2k^2}\bar{\bar{F}}_1 - 2(1 - 2x_0)\bar{\bar{F}}_1 + \frac{6\beta^4 + 3\beta^2 - 1}{4}\bar{\bar{F}}_2 - \right.$$

$$\left.\frac{2(2\beta^2 + 1)x_0}{\beta^2}\bar{\bar{F}}_2 - \frac{6\beta^2 + 5}{\beta^2}\bar{\bar{F}}_3\right] \tag{41c}$$

$$L_4 = \frac{4}{\beta\pi}\left\{\frac{1}{k}\left[(2 - x_0)\,\overline{\overline{F}}_1 - \frac{3\beta^2 + 1}{\beta^2}\,\overline{\overline{F}}_2\right] + \frac{M^2k}{3\beta^6}\left[3(1 - \beta^2 + 2\beta^4 + 2\beta^2x_0 - 4\beta^4x_0)\,\overline{\overline{F}}_2 - 6\beta^4\,(4 - 3x_0)\,\overline{\overline{F}}_3 - (1 - 7\beta^2 - 20\beta^4)\overline{\overline{F}}_4\right]\right\} \qquad (41d)$$

where $\overline{\overline{F}}_n$ $(n = 1, 2, 3, 4)$, given in the appendix, is obtained from F_n, equation (28), when $x = 2b$. For any section between the point where the Mach line intersects the trailing edge and the midspan, or where $\frac{1}{\beta} \leqq y \leqq \frac{A}{2}$, the components of section forces are:

$$\left.\begin{aligned}
L_1 &= \frac{1}{\beta^3} \\
L_2 &= \frac{1}{\beta k} - \frac{M^2k}{\beta^5} \\
L_3 &= \frac{1}{\beta k^2} - \frac{1}{3\beta^5}\left[(3 + \beta^2) + 6\beta^2x_0\right] \\
L_4 &= \frac{1}{\beta^3k}\left[(\beta^2 - 1) - 2\beta^2x_0\right] + \frac{M^2k}{6\beta^7}\left(5 + \beta^2 + 12\beta^2x_0\right)
\end{aligned}\right\} \qquad (42)$$

As a check on the results in equations (41) and (42) the expressions in equations (41) reduce to those in equations (42) when $y = \frac{1}{\beta}$.

The components of section moment coefficients for case 1 are as follows: For $0 < y < \frac{1}{\beta}$,

$$M_1 = -\frac{4}{\beta\pi}\left[(1 - 2x_0)\,\overline{\overline{F}}_1 + 2x_0\,\frac{2\beta^2 + 1}{\beta^2}\,\overline{\overline{F}}_2 - \frac{3\beta^2 + 1}{\beta^2}\,\overline{\overline{F}}_3\right] \qquad (43a)$$

$$M_2 = \frac{4}{\beta\pi}\left\{\frac{1}{k}(\overline{\overline{F}}_2 - x_0\overline{\overline{F}}_1) + \frac{M^2k}{\beta^2}\left[\frac{(2\beta^2 - 1)(1 - 2x_0)}{\beta^2}\,\overline{\overline{F}}_2 + 6x_0\overline{\overline{F}}_3 - \frac{4\beta^2 + 1}{\beta^2}\,\overline{\overline{F}}_4\right]\right\} \qquad (43b)$$

$$M_3 = \frac{4}{\beta\pi}\left[\frac{1}{k^2}\left(\overline{\overline{F}}_2 - x_0\overline{\overline{F}}_1\right) - \frac{4(1 - 3x_0 + 3x_0^2)}{3}\overline{\overline{F}}_1 + \right.$$

$$\frac{\left(6\beta^4 + 3\beta^2 - 1\right)\left(1 - 2x_0\right) + 4\beta^2 x_0^2\left(1 + 2\beta^2\right)}{\beta^4}\overline{\overline{F}}_2 +$$

$$\left.\frac{6\left(1 + \beta^2\right)}{\beta^2}x_0\overline{\overline{F}}_3 - \frac{20\beta^4 + 21\beta^2 + 3}{3\beta^4}\overline{\overline{F}}_4\right] \tag{43c}$$

$$M_4 = \frac{4}{\beta\pi}\left(\frac{2}{k}\left[\left(1 - x_0\right)^2\overline{\overline{F}}_1 + \frac{2\beta^2 + 1}{\beta^2}x_0\overline{\overline{F}}_2 - \frac{2\beta^2 + 1}{\beta^2}\overline{\overline{F}}_3\right] + \right.$$

$$\frac{4M^2k}{\beta^2}\left\{\frac{1}{6\beta^4}\left[\left(4\beta^4 - 2\beta^2 + 1\right)\left(1 - 3x_0 + 3x_0^2\right) + 1 - 3x_0^2\right]\overline{\overline{F}}_2 - \right.$$

$$\left(2 - 4x_0 + 3x_0^2\right)\overline{\overline{F}}_3 - \frac{x_0}{6\beta^4}\left(8\beta^4 + 4\beta^2 - 1\right)\overline{\overline{F}}_4 +$$

$$\left.\left.\frac{1}{3\beta^2}\left(5\beta^2 + 3\right)\overline{\overline{F}}_5\right\}\right) \tag{43d}$$

and for $\frac{1}{\beta} \leqq y \leqq \frac{A}{2}$,

$$\left.\begin{aligned}
M_1 &= \frac{2}{3\beta^3}(2 - 3x_0) \\
M_2 &= \frac{1}{\beta k}(1 - 2x_0) - \frac{M^2k}{2\beta^5}(3 - 4x_0) \\
M_3 &= \frac{1}{\beta k^2}(1 - 2x_0) - \frac{1}{2\beta^5}(3 + \beta^2 - 4x_0 + 4\beta^2x_0 - 8\beta^2x_0^2) \\
M_4 &= \frac{2}{3\beta^3k}\left[2(\beta^2 - 1) - 3x_0(2\beta^2 - 1) + 6\beta^2x_0^2\right] + \\
&\quad \frac{M^2k}{15\beta^7}\left[4(5 + \beta^2) + 5x_0(8\beta^2 - 5) - 60\beta^2x_0^2\right]
\end{aligned}\right\} \tag{44}$$

The expressions in equations (43) reduce to those in equations (44) when $y = \frac{1}{\beta}$. The expressions in equations (42) and (44) correspond to the more exact two-dimensional components of force and moment coefficients derived in reference 4. For values of $k < 0.1$ these expressions yield, for the most part, values that are in good agreement with those that may be calculated from the tables in reference 4.

Case 2 (see fig. 3(b)): For any section between the tip at $y = 0$ and the point where the Mach line from the tip at $y = 2s$ intersects the trailing edge $\left(\text{or where } 0 < y \leqq A - \frac{1}{\beta}\right)$, the components of section force coefficients are given by equations (41) and the components of section moment coefficients, by equations (43). For any section between the point where the Mach line from the tip at $y = 2s$ intersects the trailing edge and the midspan $\left(\text{or where } A - \frac{1}{\beta} < y \leqq \frac{A}{2}\right)$, the components of section force coefficients are

$$L_1 = -\left\{\frac{4}{\beta\pi}\left[(\overline{\overline{F}}_1 + \overline{\overline{G}}_1) - \frac{1 + 2\beta^2}{\beta^2}(\overline{\overline{F}}_2 + \overline{\overline{G}}_2)\right] + \frac{1}{\beta^3}\right\} \tag{45a}$$

$$L_2 = \frac{4}{\beta\pi}\left\{\frac{1}{2k}(\overline{\overline{F}}_1 + \overline{\overline{G}}_1) + \frac{M^2k}{\beta^4}\left[(2\beta^2 - 1)(\overline{\overline{F}}_2 + \overline{\overline{G}}_2) - 3\beta^2(\overline{\overline{F}}_3 + \overline{\overline{G}}_3)\right]\right\} - \left(\frac{1}{\beta k} - \frac{M^2k}{\beta^5}\right) \tag{45b}$$

$$L_3 = \frac{4}{\beta\pi}\left[\frac{1}{2k^2}(\overline{\overline{F}}_1 + \overline{\overline{G}}_1) - 2(1 - 2x_0)(\overline{\overline{F}}_1 + \overline{\overline{G}}_1) + \frac{6\beta^4 + 3\beta^2 - 1}{\beta^4}(\overline{\overline{F}}_2 + \overline{\overline{G}}_2) - \frac{2(2\beta^2 + 1)x_0}{\beta^2}(\overline{\overline{F}}_2 + \overline{\overline{G}}_2) - \frac{6\beta^2 + 5}{\beta^2}(\overline{\overline{F}}_3 + \overline{\overline{G}}_3)\right] - \left\{\frac{1}{\beta k^2} - \frac{1}{3\beta^5}\left[(3 + \beta^2) + 6\beta^2x_0\right]\right\} \tag{45c}$$

$$L_4 = \frac{4}{\beta\pi}\left\{\frac{1}{k}\left[(2 - x_0)(\bar{\bar{F}}_1 + \bar{\bar{G}}_1) - \frac{3\beta^2 + 1}{\beta^2}(\bar{\bar{F}}_2 + \bar{\bar{G}}_2)\right] + \frac{M^2k}{3\beta^6}\left[3(1 - \beta^2 + 2\beta^4 + 2\beta^2x_0 - 4\beta^4x_0)(\bar{\bar{F}}_2 + \bar{\bar{G}}_2) - 6\beta^4(4 - 3x_0)(\bar{\bar{F}}_3 + \bar{\bar{G}}_3) - (1 - 7\beta^2 - 20\beta^4)(\bar{\bar{F}}_4 + \bar{\bar{G}}_4)\right]\right\} - \left\{\frac{1}{\beta^3k}\left[(\beta^2 - 1) - 2\beta^2x_0\right] + \frac{M^2k}{6\beta^7}\left(5 + \beta^2 + 12\beta^2x_0\right)\right\} \tag{45d}$$

where $\bar{\bar{G}}_n$ $(n = 1, 2, 3, 4)$, given in the appendix, is obtained from G_n, equation (29), when $x = 2b$. The corresponding components of section moment coefficients are

$$M_1 = -\left\{\frac{4}{\beta\pi}\left[(1 - 2x_0)(\bar{\bar{F}}_1 + \bar{\bar{G}}_1) + \frac{2x_0(2\beta^2 + 1)}{\beta^2}(\bar{\bar{F}}_2 + \bar{\bar{G}}_2)\right] - \frac{3\beta^2 + 1}{\beta^2}(\bar{\bar{F}}_3 + \bar{\bar{G}}_3) + \frac{2}{3\beta^3}(2 - 3x_0)\right\} \tag{46a}$$

$$M_2 = \frac{4}{\beta\pi}\left\{\frac{1}{k}\left[(\bar{\bar{F}}_2 + \bar{\bar{G}}_2) - x_0(\bar{\bar{F}}_1 + \bar{\bar{G}}_1)\right] + \frac{M^2k}{\beta^2}\left[\frac{(2\beta^2 - 1)(1 - 2x_0)}{\beta^2}(\bar{\bar{F}}_2 + \bar{\bar{G}}_2) + 6x_0(\bar{\bar{F}}_3 + \bar{\bar{G}}_3) - \frac{4\beta^2 + 1}{\beta^2}(\bar{\bar{F}}_4 + \bar{\bar{G}}_4)\right]\right\} - \left[\frac{1}{\beta k}(1 - 2x_0) - \frac{M^2k}{2\beta^5}(3 - 4x_0)\right] \tag{46b}$$

$$M_3 = \frac{4}{\beta\pi}\left\{\frac{1}{k^2}\left[(\bar{\bar{F}}_2 + \bar{\bar{G}}_2) - x_0(\bar{\bar{F}}_1 + \bar{\bar{G}}_1)\right] - 4\,\frac{1 - 3x_0 + 3x_0^2}{3}(\bar{\bar{F}}_1 + \bar{\bar{G}}_1) + \right.$$

$$\frac{(6\beta^4 + 3\beta^2 - 1)(1 - 2x_0) + 4\beta^2 x_0^2(1 + 2\beta^2)}{\beta^4}(\bar{\bar{F}}_2 + \bar{\bar{G}}_2) +$$

$$\left.\frac{6x_0(1 + \beta^2)}{\beta^2}(\bar{\bar{F}}_3 + \bar{\bar{G}}_3) - \frac{20\beta^4 + 21\beta^2 + 3}{3\beta^4}(\bar{\bar{F}}_4 + \bar{\bar{G}}_4)\right\} -$$

$$\left\{\frac{1}{\beta k^2}(1 - 2x_0) - \frac{1}{2\beta^5}\left[(3 + \beta^2) - 4x_0(1 - \beta^2) - 8\beta^2 x_0^2\right]\right\} \tag{46c}$$

$$M_4 = \frac{4}{\beta\pi}\left(\frac{2}{k}\left[(1 - x_0)^2(\bar{\bar{F}}_1 + \bar{\bar{G}}_1) + \frac{2\beta^2 + 1}{\beta^2}x_0(\bar{\bar{F}}_2 + \bar{\bar{G}}_2) - \frac{2\beta^2 + 1}{\beta^2}(\bar{\bar{F}}_3 + \bar{\bar{G}}_3)\right] + \right.$$

$$\frac{4M^2 k}{\beta^2}\left\{\frac{1}{6\beta^4}\left[(4\beta^4 - 2\beta^2 + 1)(1 - 3x_0 + 3x_0^2) + 1 - 3x_0^2\right](\bar{\bar{F}}_2 + \bar{\bar{G}}_2) - \right.$$

$$(2 - 4x_0 + 3x_0^2)(\bar{\bar{F}}_3 + \bar{\bar{G}}_3) - \frac{x_0}{6\beta^4}(8\beta^4 + 4\beta^2 - 1)(\bar{\bar{F}}_4 + \bar{\bar{G}}_4) +$$

$$\left.\left.\frac{1}{3\beta^2}(5\beta^2 + 3)(\bar{\bar{F}}_5 + \bar{\bar{G}}_5)\right\}\right) - \left\{\frac{2}{3\beta^3 k}\left[2(\beta^2 - 1) - 3x_0(2\beta^2 - 1) + 6\beta^2 x_0^2\right] + \right.$$

$$\left.\frac{M^2 k}{15\beta^7}\left[4(5 + \beta^2) + 5x_0(8\beta^2 - 5) - 60\beta^2 x_0^2\right]\right\} \tag{46d}$$

For the limiting condition of case 2, that is, when the Mach line from one tip intersects the opposite tip at the trailing edge, or $A = \frac{1}{\beta}$ (see fig. 3(c)), the components of section force coefficients are given by equations (45) and the corresponding components of moment coefficients by equations (46).

SOME PARTICULAR CALCULATIONS AND DISCUSSIONS

From the expressions for total force and moment coefficients, equations (35) and (38), respectively, the over-all effect of aspect ratio on the magnitude of the forces and moments can be calculated for particular values of the parameters M, k, x_0, and A. Examination of these equations shows that varying some of the parameters might cause some terms in the equations to vanish and change sign. For example, if x_0 is continuously increased from some value less than 1/2 to some value greater than 1/2, the first terms in the expressions for $\overline{M_2}$ and $\overline{M_3}$ vanish at $x_0 = \frac{1}{2}$ and change sign when x_0 becomes greater than 1/2. In particular, decreasing the aspect ratio decreases the components of force and moment coefficients $\overline{L_1}$, $\overline{L_2}$, $\overline{L_3}$, $\overline{M_1}$, $\overline{M_2}$, and $\overline{M_3}$ but increases the two important components $\overline{L_4}$ and $\overline{M_4}$.

Although the effect of aspect ratio may change considerably with only a small change in any one (or more) of the parameters M, k, and x_0, some insight into what the over-all effect might be can be gained from calculations of all the components of total force and moment coefficients for various values of M and A and fixed values of the parameters k and x_0. Results of such a set of calculations are presented in figures 4 to 7.

In figures 4 and 5 the components of total force and moment coefficients for various values of A and for $x_0 = 0.4$ and $k = 0.02$ are plotted as functions of M^2. The curves in these figures calculated for infinite aspect ratio correspond to the two-dimensional results of reference 4. The dashed curves represent calculations for aspect ratio and Mach number combinations that cause the Mach lines from one tip to intersect the opposite tip at the trailing edge so that along the dashed curves the aspect ratio is not constant but varies with M^2 according to the previously given expression

$$A = \frac{1}{\beta} = \frac{1}{\sqrt{M^2 - 1}}$$

The difference, at any value of M^2, between the dashed curves and the curves corresponding to infinite aspect ratio in figures 4 and 5 is, therefore, for the chosen values of k and x_0, the maximum effect of aspect ratio on the components of total force and moment coefficients for a rectangular wing under the restrictions of the foregoing analysis. It will be noted in figures 4 and 5 that, when the aspect ratio is small, the deviation of the three-dimensional results from two-dimensional results may be quite large.

In figures 6 and 7 the components of the total force and moment coefficients are plotted as functions of aspect ratio for $x_0 = 0.4$, $k = 0.02$, and some particular values of M. It will be noted in these figures that all the components of force and moment coefficients undergo rapid changes with respect to varying aspect ratio when A becomes less than 4 or 5. It may be remarked that the directions of the changes with respect to aspect ratio appear to be such that they would have favorable effects on the flutter characteristics of a wing.

The spanwise distribution of the components of section force and moment coefficients computed from equations (41) to (44) for $A = 4$, $x_0 = 0.4$, $k = 0.02$, and $M = 2$ are plotted in figures 8 and 9. The portions of the curves in these figures corresponding to values of y in the range $\frac{1}{\beta} \leqq y \leqq A - \frac{1}{\beta}$ are the two-dimensional values and the effect of aspect ratio may be noted in the tip regions, $0 \leqq y \leqq \frac{1}{\beta}$ and $A - \frac{1}{\beta} \leqq y \leqq A$, as deviations from these two-dimensional values.

In conclusion, it may be stated that, in regard to the effect of aspect ratio on supersonic flutter, an important item that has not been discussed herein but can be studied for any particular case with the aid of equations (33) and (36) is the change in center of pressure, associated with prescribed motions of the wing, with change in aspect ratio. An investigation to find the effect that thickness might have on the center-of-pressure location is also needed. An extension of the foregoing analysis to include the effect of an aileron as an additional degree of freedom would follow in a straightforward manner.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Air Force Base, Va., January 6, 1950

APPENDIX

SOME INTEGRATED VALUES OF F_n, G_n, $\bar{\bar{F}}_n$, $\bar{\bar{G}}_n$, AND OTHER RELATED FUNCTIONS

Values of F_n and G_n.- The values of the functions F_n, equation (28), and G_n, equation (29), for the first few values of n are as follows:

$$F_n = \int_0^x x^{n-1} \sin^{-1}\sqrt{\beta y/x}\, dx \qquad (n = 1, 2, 3, \ldots)$$

$$F_1 = \sqrt{\beta y(x - \beta y)} + x \sin^{-1}\sqrt{\beta y/x}$$

$$F_2 = \frac{x + 2\beta y}{6}\sqrt{\beta y(x - \beta y)} + \frac{x^2}{2}\sin^{-1}\sqrt{\beta y/x}$$

$$F_3 = \frac{3x^2 + 4\beta yx + 8\beta^2 y^2}{45}\sqrt{\beta y(x - \beta y)} + \frac{x^3}{3}\sin^{-1}\sqrt{\beta y/x}$$

$$F_4 = \frac{5x^3 - 6\beta yx^2 + 8\beta^2 y^2 x + 16\beta^3 y^3}{140}\sqrt{\beta y(x - \beta y)} + \frac{x^4}{4}\sin^{-1}\sqrt{\beta y/x}$$

$$F_5 = \frac{35x^4 + 40\beta yx^3 + 48\beta^2 y^2 x^2 + 64\beta^3 y^3 x + 128\beta^4 y^4}{1575}\sqrt{\beta y(x - \beta y)} + \frac{x^5}{5}\sin^{-1}\sqrt{\beta y/x}$$

$$G_n = \int_0^x x^{n-1} \sin^{-1}\sqrt{\frac{\beta(2s - y)}{x}}\, dx \qquad (n = 1, 2, 3, \ldots)$$

$$G_1 = \sqrt{\beta(2s - y)\left[x - \beta(2s - y)\right]} + x \sin^{-1}\sqrt{\frac{\beta(2s - y)}{x}}$$

$$G_2 = \frac{x + 2\beta(2s - y)}{6}\sqrt{\beta(2s - y)\left[x - \beta(2s - y)\right]} + \frac{x^2}{2}\sin^{-1}\sqrt{\frac{\beta(2s - y)}{x}}$$

$$G_3 = \frac{3x^2 + 4\beta x(2s - y) + 8\beta^2(2s - y)^2}{45}\sqrt{\beta(2s - y)\left[x - \beta(2s - y)\right]} + \frac{x^3}{3}\sin^{-1}\sqrt{\frac{\beta(2s - y)}{x}}$$

$$G_4 = \frac{5x^3 + 6x^2\beta(2s - y) + 8x\beta^2(2s - y)^2 + 16\beta^3(2s - y)^3}{140}\sqrt{\beta(2s - y)\left[x - \beta(2s - y)\right]} +$$
$$\frac{x^4}{4}\sin^{-1}\sqrt{\frac{\beta(2s - y)}{x}}$$

$$G_5 = \frac{35x^4 + 40x^3\beta(2s - y) + 48x^2\beta^2(2s - y)^2 + 64x\beta^3(2s - y)^3 + 128\beta^4(2s - y)^4}{1575}\sqrt{\beta(2s - y)\left[x - \beta(2s - y)\right]} +$$
$$\frac{x^5}{5}\sin^{-1}\sqrt{\frac{\beta(2s - y)}{x}}$$

Values of $\overline{\overline{F}}_n$ and $\overline{\overline{G}}_n$.- The following expressions define the fuctions $\overline{\overline{F}}_n$ and $\overline{\overline{G}}_n$ appearing in equations (41), (43), (45), and (46) in the body of the paper. In these expressions the variable y has been referred to the chord length $2b$; that is, $y/2b$ has been replaced by y and in the expressions for $\overline{\overline{G}}_n$ the ratio s/b has been replaced by A:

$$\overline{\overline{F}}_1 = \sqrt{\beta y(1 - \beta y)} + \sin^{-1}\sqrt{\beta y}$$

$$\overline{\overline{F}}_2 = \frac{1 + 2\beta y}{6}\sqrt{\beta y(1 - \beta y)} + \frac{1}{2}\sin^{-1}\sqrt{\beta y}$$

$$\overline{\overline{F}}_3 = \frac{3 + 4\beta y + 8\beta^2 y^2}{45}\sqrt{\beta y(1 - \beta y)} + \frac{1}{3}\sin^{-1}\sqrt{\beta y}$$

$$\overline{\overline{F}}_4 = \frac{5 + 6\beta y + 8\beta^2 y^2 + 16\beta^3 y^3}{140}\sqrt{\beta y(1 - \beta y)} + \frac{1}{4}\sin^{-1}\sqrt{\beta y}$$

$$\overline{\overline{F}}_5 = \frac{35 + 40\beta y + 48\beta^2 y^2 + 64\beta^3 y^3 + 128\beta^4 y^4}{1575}\sqrt{\beta y(1 - \beta y)} + \frac{1}{5}\sin^{-1}\sqrt{\beta y}$$

$$\overline{\overline{G}}_1 = \sqrt{\beta(A - y)\left[1 - \beta(A - y)\right]} + \sin^{-1}\sqrt{\beta(A - y)}$$

$$\overline{\overline{G}}_2 = \frac{1 + 2\beta(A - y)}{6}\sqrt{\beta(A - y)\left[1 - \beta(A - y)\right]} + \frac{1}{2}\sin^{-1}\sqrt{\beta(A - y)}$$

$$\overline{\overline{G}}_3 = \frac{3 + 4\beta(A - y) + 8\beta^2(A - y)^2}{45}\sqrt{\beta(A - y)\left[1 - \beta(A - y)\right]} + \frac{1}{3}\sin^{-1}\sqrt{\beta(A - y)}$$

$$\overline{\overline{G}}_4 = \frac{5 + 6\beta(A - y) + 8\beta^2(A - y)^2 + 16\beta^3(A - y)^3}{140}\sqrt{\beta(A - y)\left[1 - \beta(A - y)\right]} + \frac{1}{4}\sin^{-1}\sqrt{\beta(A - y)}$$

$$\overline{\overline{G}}_5 = \frac{35 + 40\beta(A - y) + 48\beta^2(A - y)^2 + 64\beta^3(A - y)^3 + 128\beta^4(A - y)^4}{1575}\sqrt{\beta(A - y)\left[1 - \beta(A - y)\right]} +$$

$$\frac{1}{5}\sin^{-1}\sqrt{\beta(A - y)}$$

<u>Some integral relations for F_n and G_n.-</u>

$$\int_0^x F_n\,dx = xF_n - F_{n+1}$$

$$\int_0^x xF_n\,dx = \frac{1}{2}\left(x^2F_n - F_{n+2}\right)$$

$$\int_0^x x^2F_n\,dx = \frac{1}{3}\left(x^3F_n + F_{n+3}\right)$$

$$\int_0^x (x - x_0)F_n\,dx = \frac{x^2 - 2xx_0}{2}F_n + x_0F_{n+1} - \frac{1}{2}F_{n+2}$$

$$\int_0^x (x - x_0)^2 F_n\,dx = \frac{x^3 - 3x_0x^2 + 3x_0^2x}{3}F_n - x_0^2F_{n+1} + x_0F_{n+2} - \frac{1}{3}F_{n+3}$$

Corresponding integral relations for G_n may be obtained from these relations by simply replacing F by G.

Integral relations for $\bar{\bar{F}}_n$.- Integral relations for $\bar{\bar{F}}_n$ that may be used in calculating total forces and moments from sectional forces and moments are as follows:

$$2b\int_0^{1/\beta} \bar{\bar{F}}_1\,dy = \frac{3b\pi}{4\beta}$$

$$2b\int_0^{1/\beta} \bar{\bar{F}}_2\,dy = \frac{b\pi}{3\beta}$$

$$2b\int_0^{1/\beta} \bar{\bar{F}}_3\,dy = \frac{5b\pi}{24\beta}$$

$$2b\int_0^{1/\beta} \bar{\bar{F}}_4\,dy = \frac{3b\pi}{20\beta}$$

$$2b\int_0^{1/\beta} \bar{\bar{F}}_5\,dy = \frac{7b\pi}{60\beta}$$

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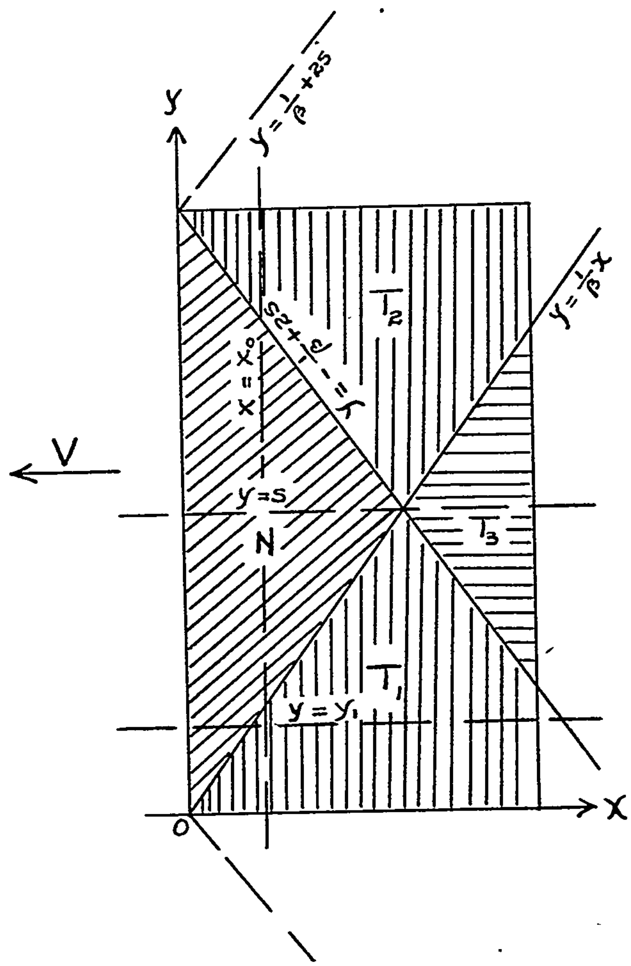
2. Garrick, I. E., and Rubinow, S. I.: Theoretical Study of Air Forces on an Oscillating or Steady Thin Wing in a Supersonic Main Stream. NACA Rep. 872, 1947.

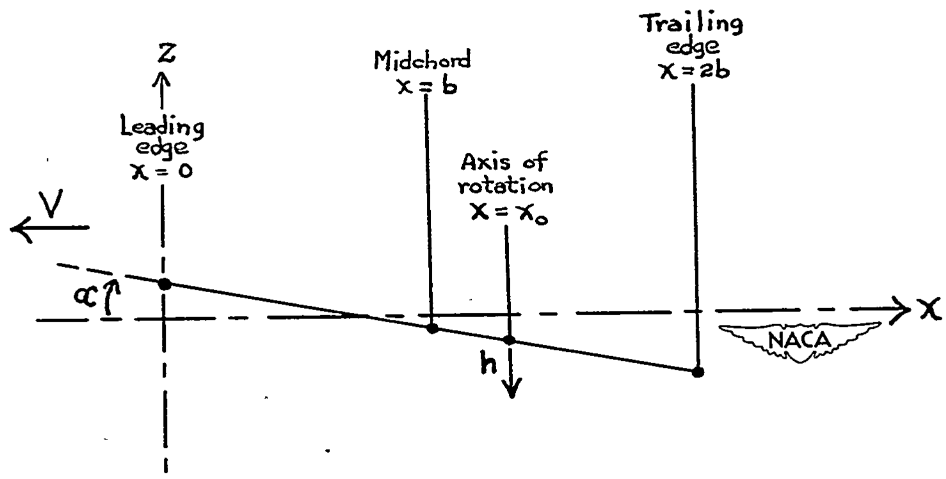
3. Miles, John W.: The Oscillating Rectangular Airfoil at Supersonic Speeds. TM RRB-15, U. S. Naval Ordnance Test Station, Inyokern, Calif., June 1, 1949.

4. Garrick, I. E., and Rubinow, S. I.: Flutter and Oscillating Air-Force Calculations for an Airfoil in a Two-Dimensional Supersonic Flow. NACA Rep. 846, 1946.

5. Snow, R. M., and Bonney, E. A.: Aerodynamic Characteristics of Wings at Supersonic Speeds. Bumblebee Rep. No. 55, The Johns Hopkins Univ., Appl. Phys. Lab., March 1947.

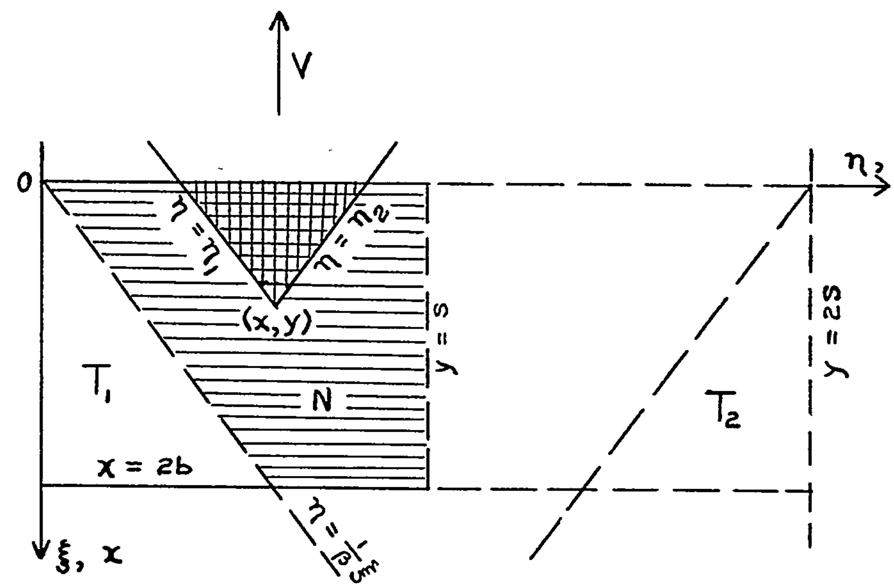


(a) Plan form (xy-plane).

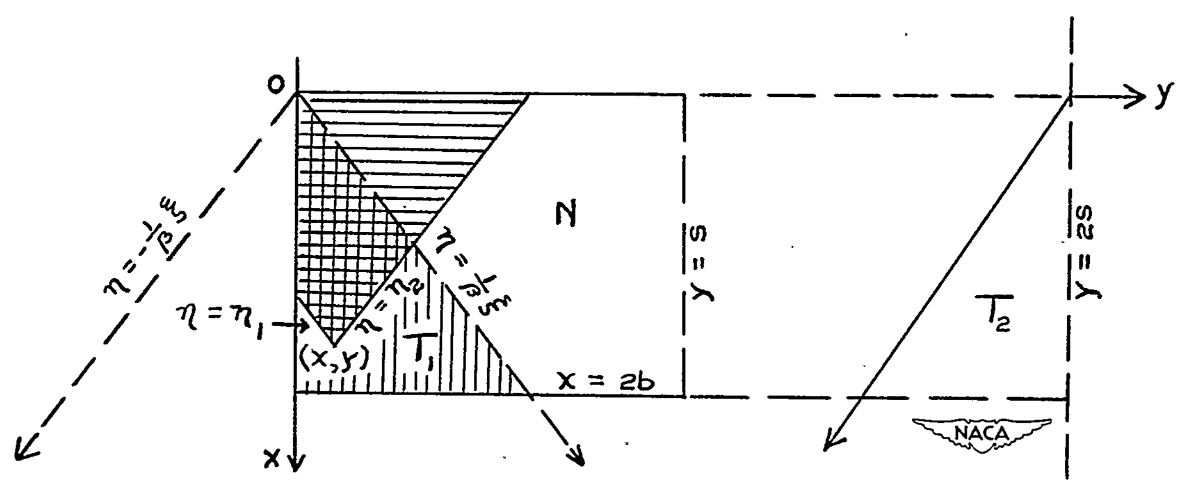


(b) Section $y = y_1$ (xz-plane).

Figure 1.- Sketch illustrating chosen coordinate system and the two degrees of freedom α and h.

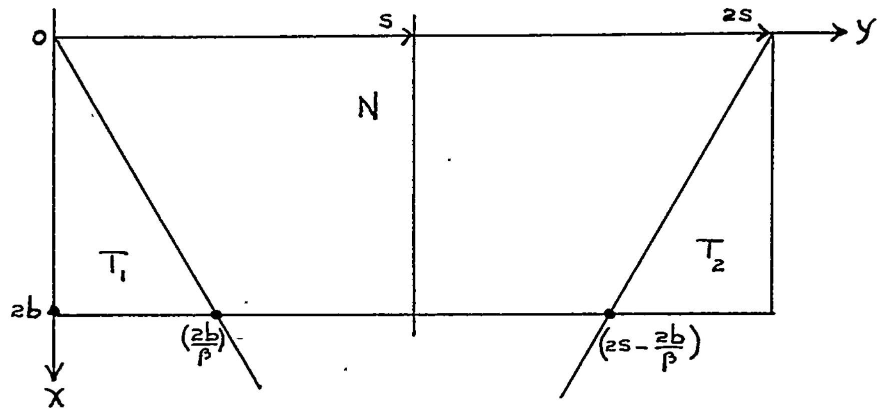


(a) Purely supersonic region.

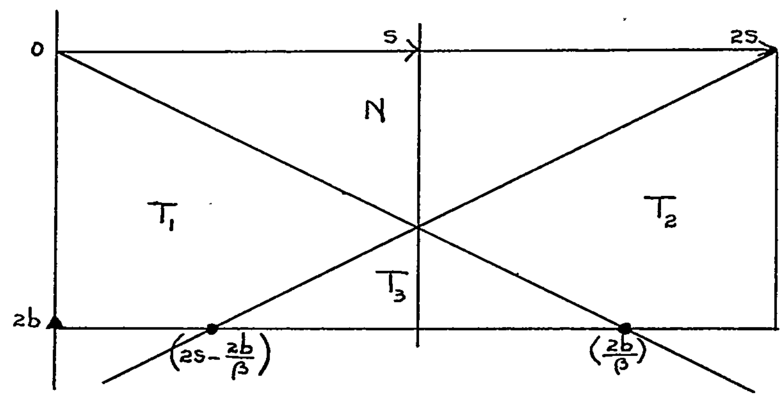


(b) Mixed supersonic region.

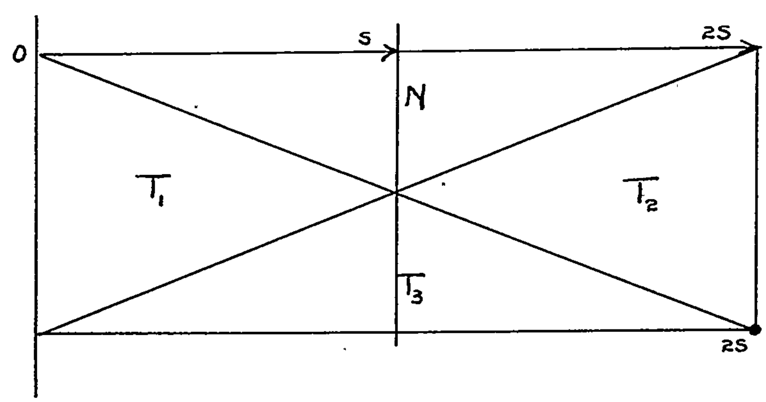
Figure 2.- Sketch illustrating areas of integration for "purely supersonic" and "mixed supersonic" regions of flow.



(a) Mach lines from tips do not intersect on wing.

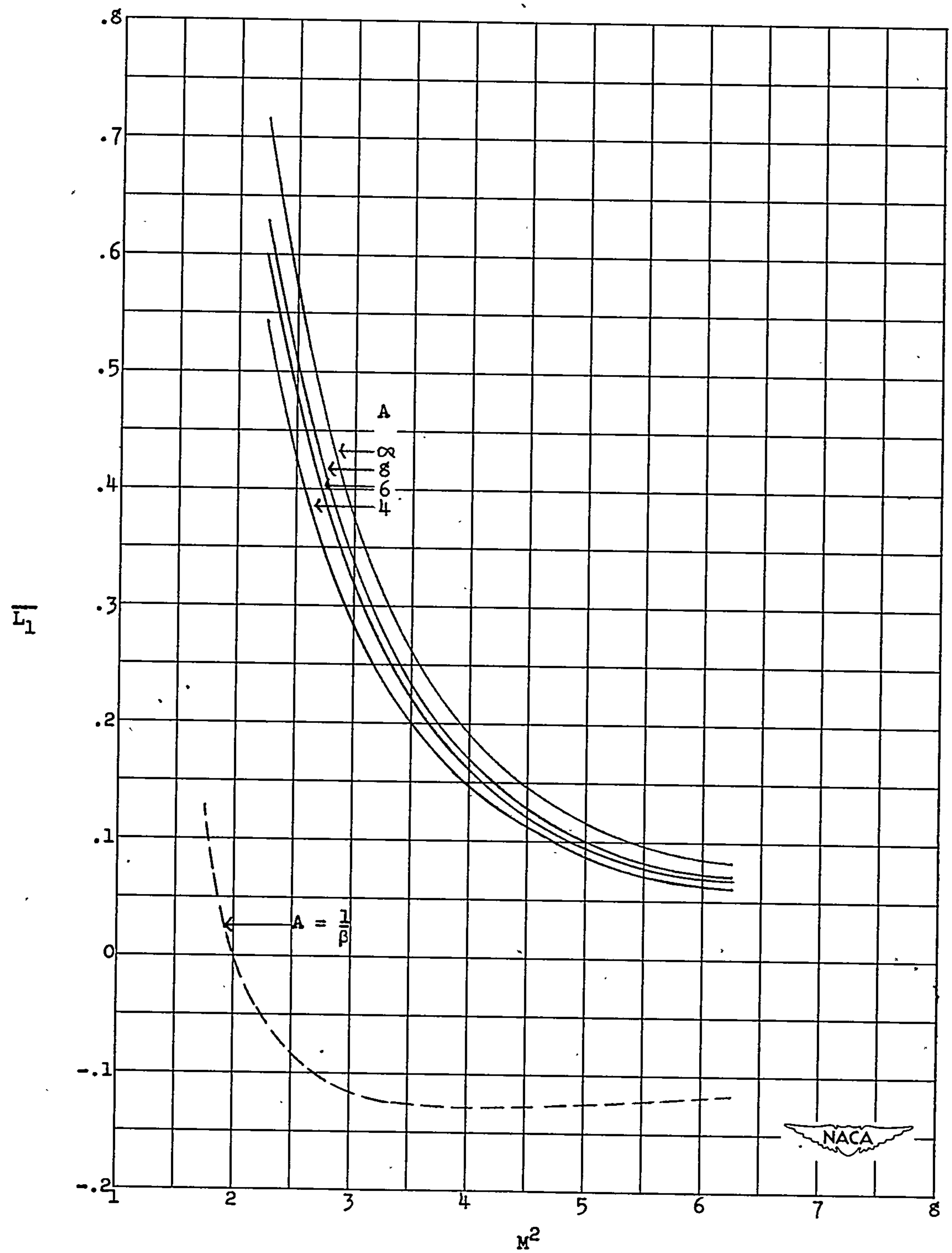


(b) Mach lines from tips intersect on wing but Mach line from one tip does not intersect opposite tip.



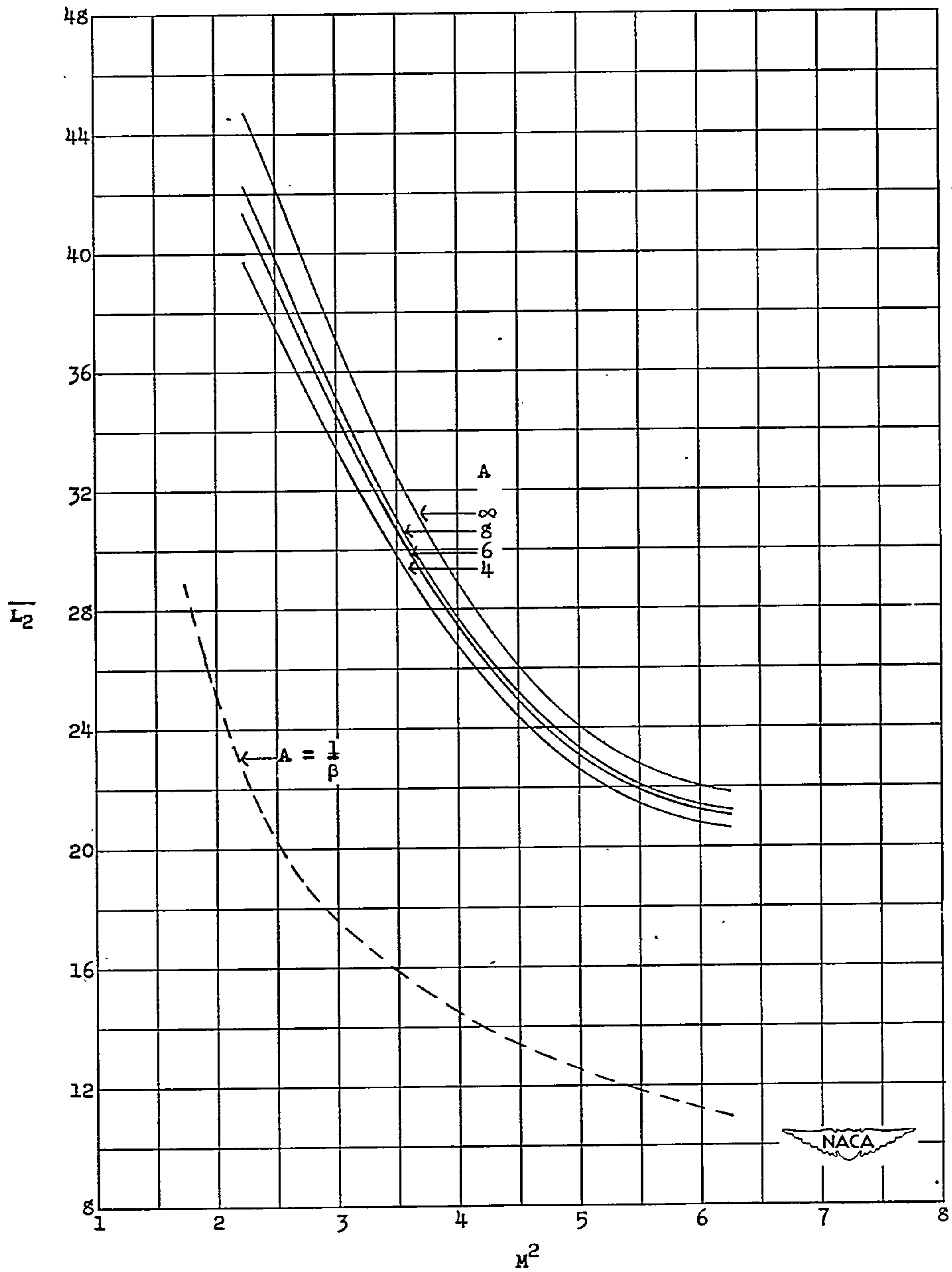
(c) Mach lines from tips intersect on wing and Mach line from one tip intersects opposite tip at trailing edge.

Figure 3.- Sketch illustrating different Mach line locations accounted for in analysis.



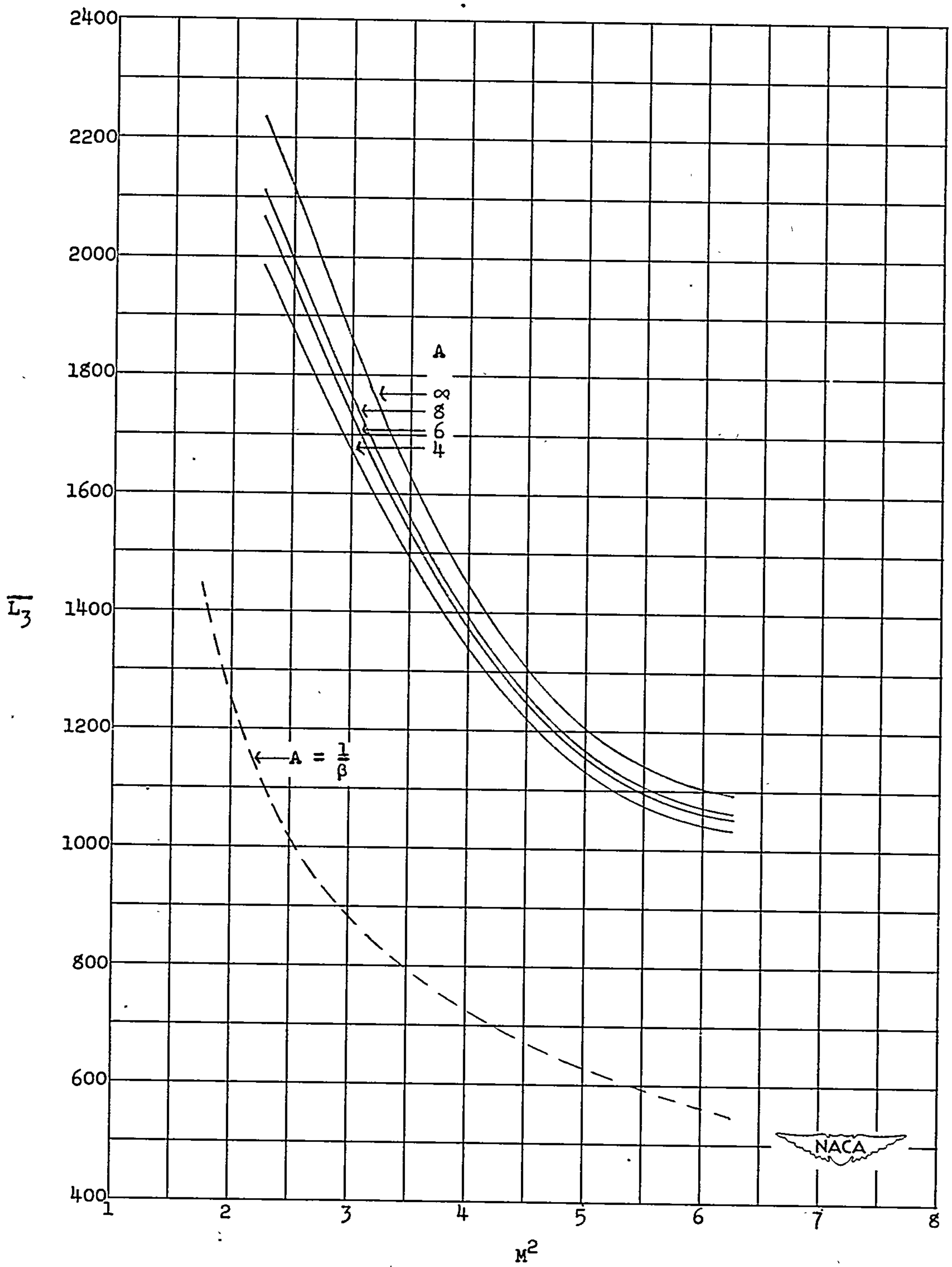
(a) $\overline{L_1}$.

Figure 4.- Components of total force coefficients as functions of M^2 for $x_0 = 0.4$, $k = 0.02$, and various values of A.



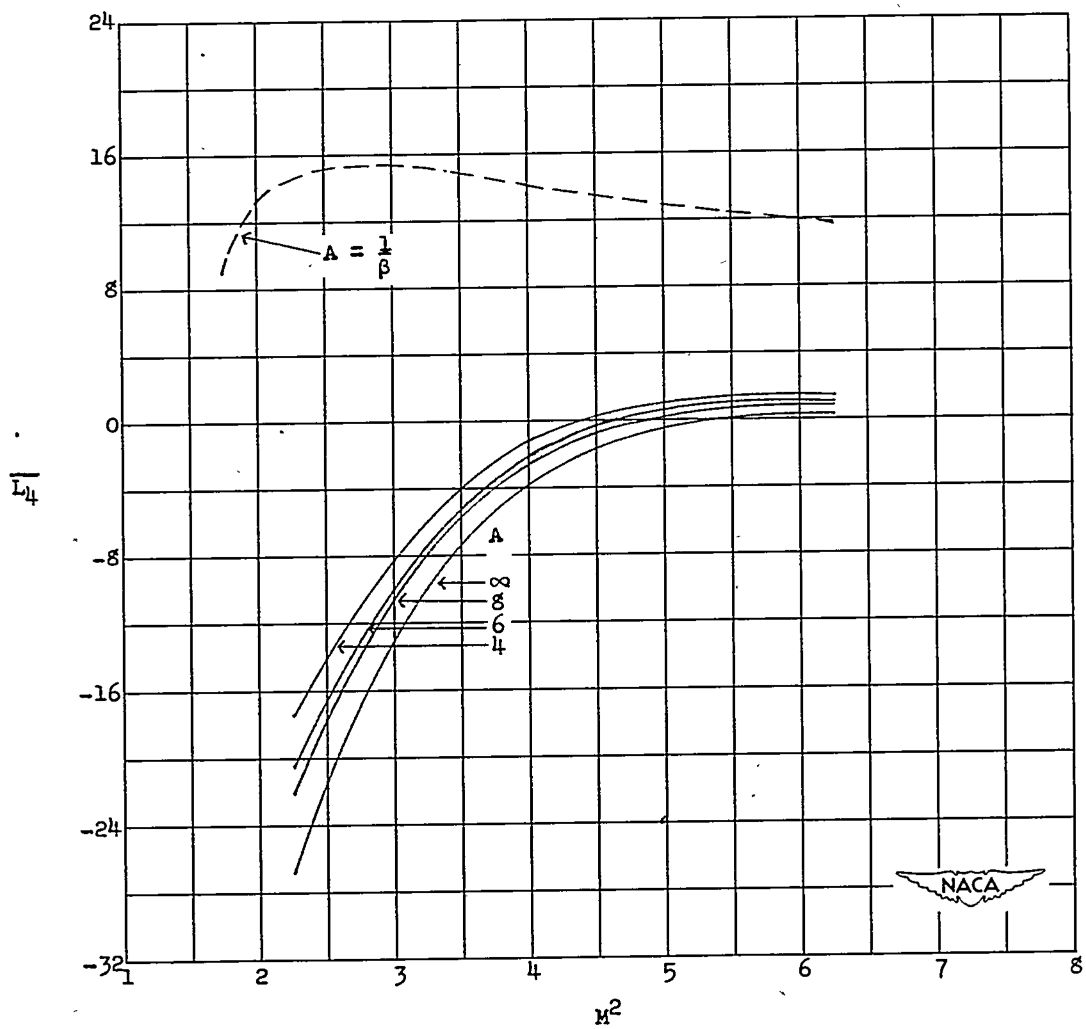
(b) $\overline{L_2}$.

Figure 4.- Continued.



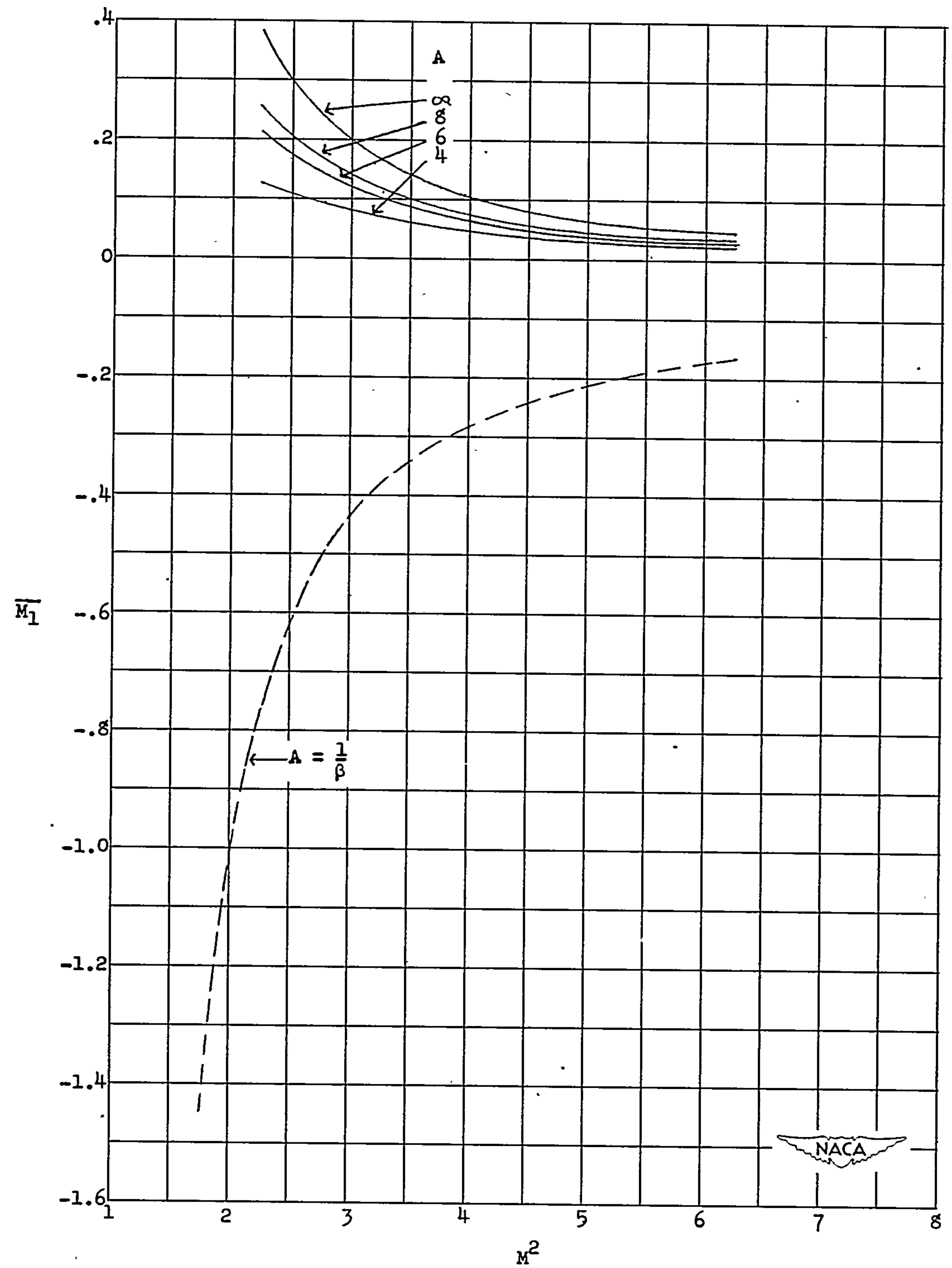
(c) $\overline{L_3}$.

Figure 4.- Continued.



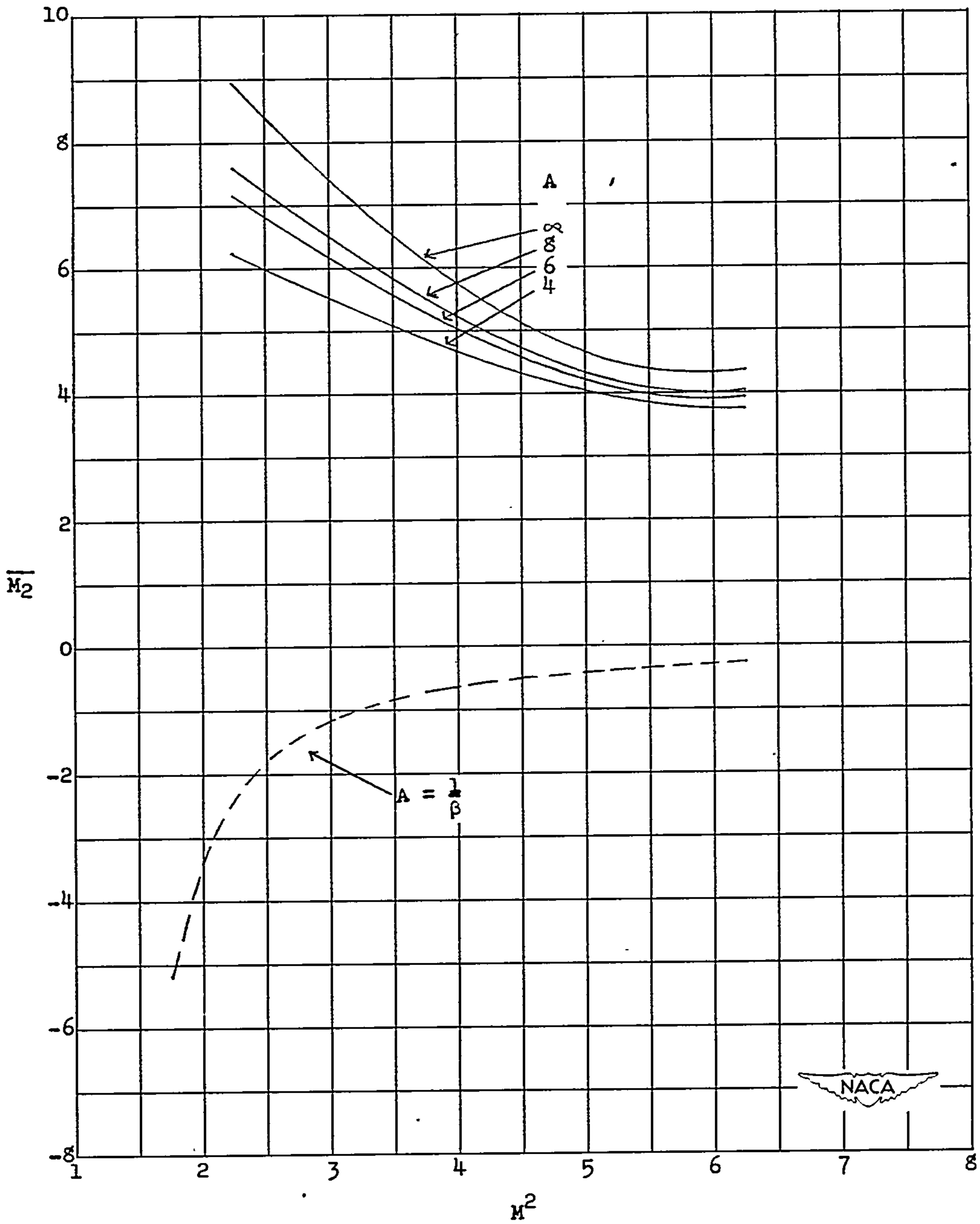
(d) $\overline{L_4}$.

Figure 4.- Concluded.



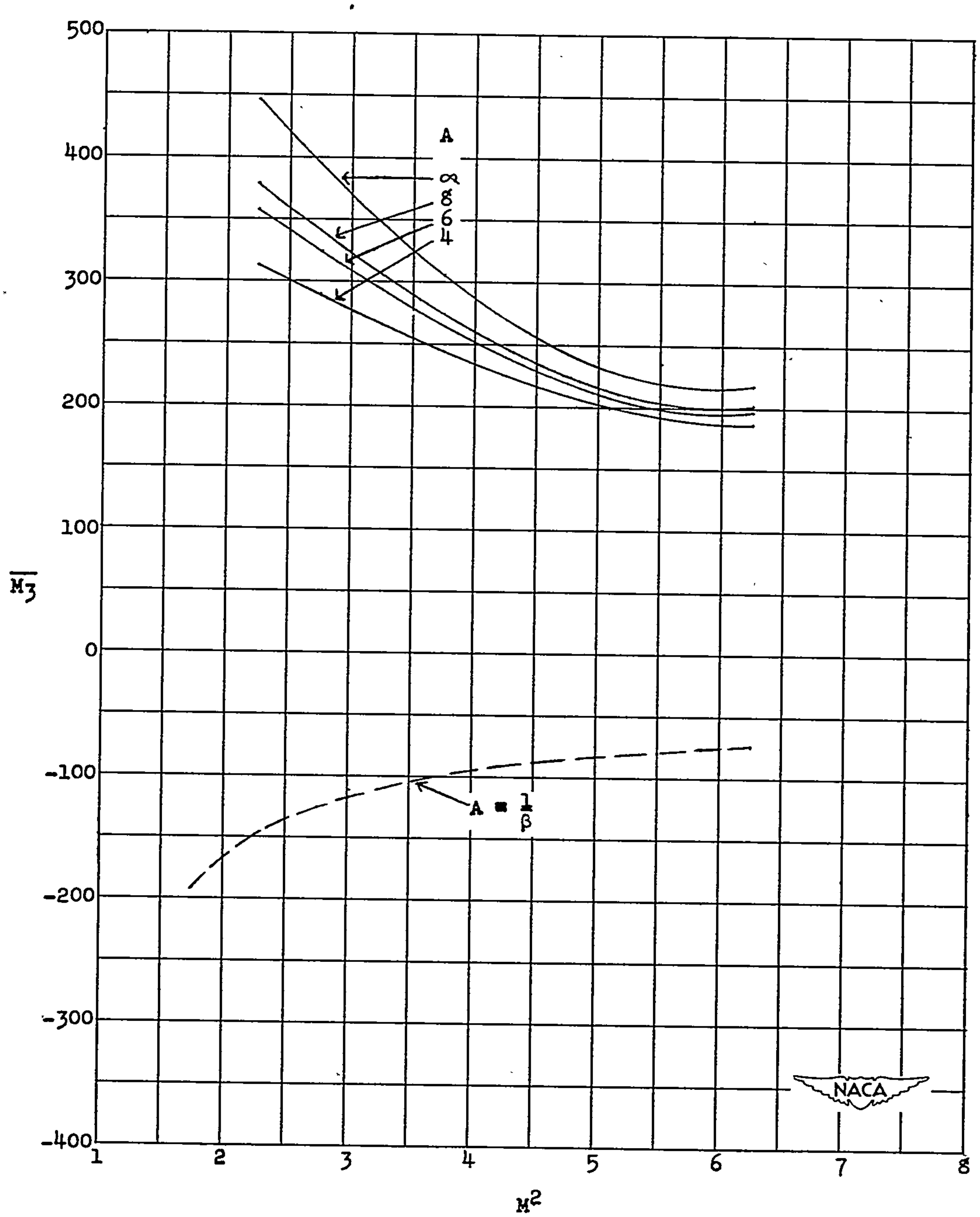
(a) $\overline{M_1}$.

Figure 5.- Components of total moment coefficients as functions of M^2 for $x_0 = 0.4$, $k = 0.02$, and various values of A.



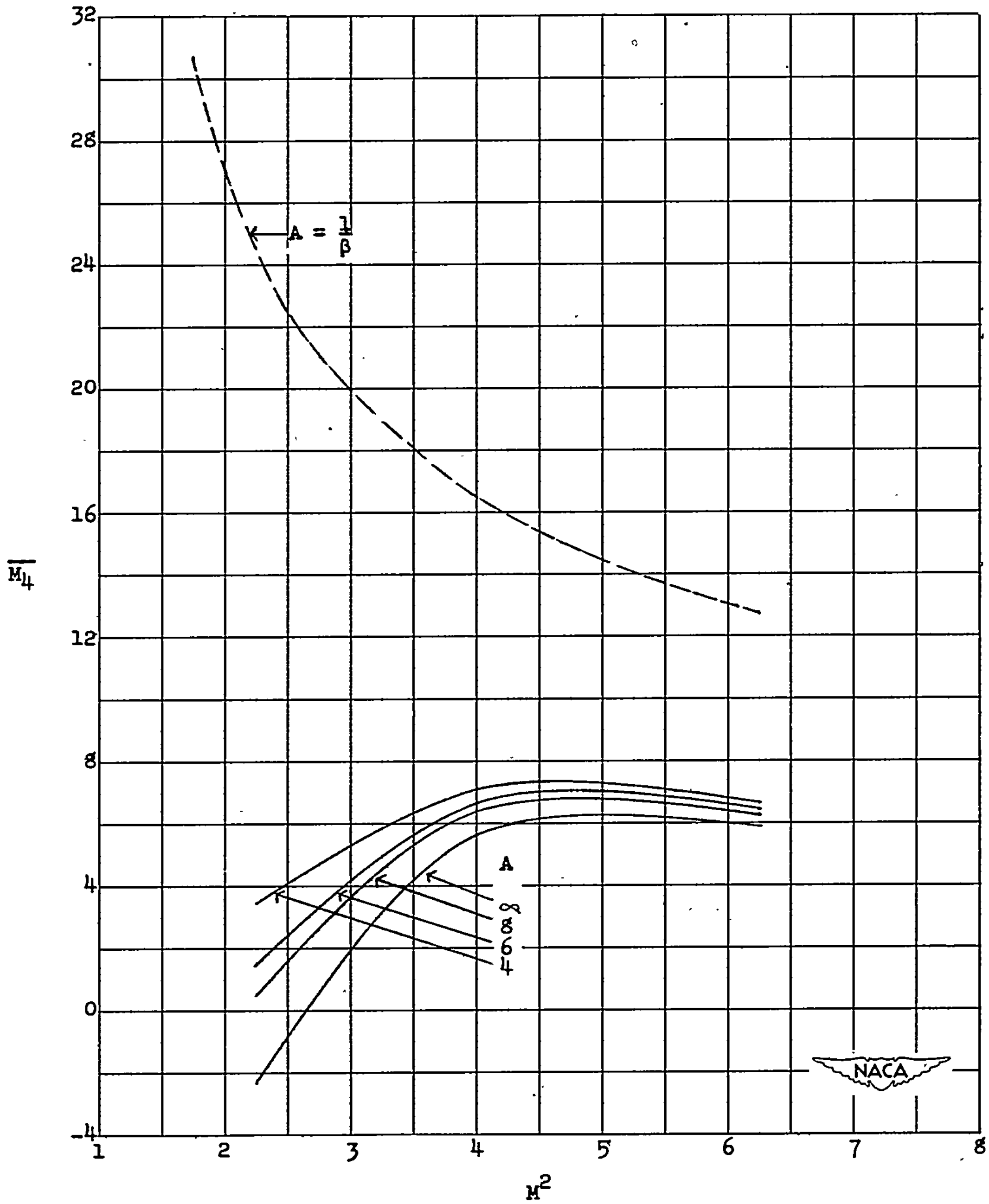
(b) $\overline{M_2}$.

Figure 5.- Continued.



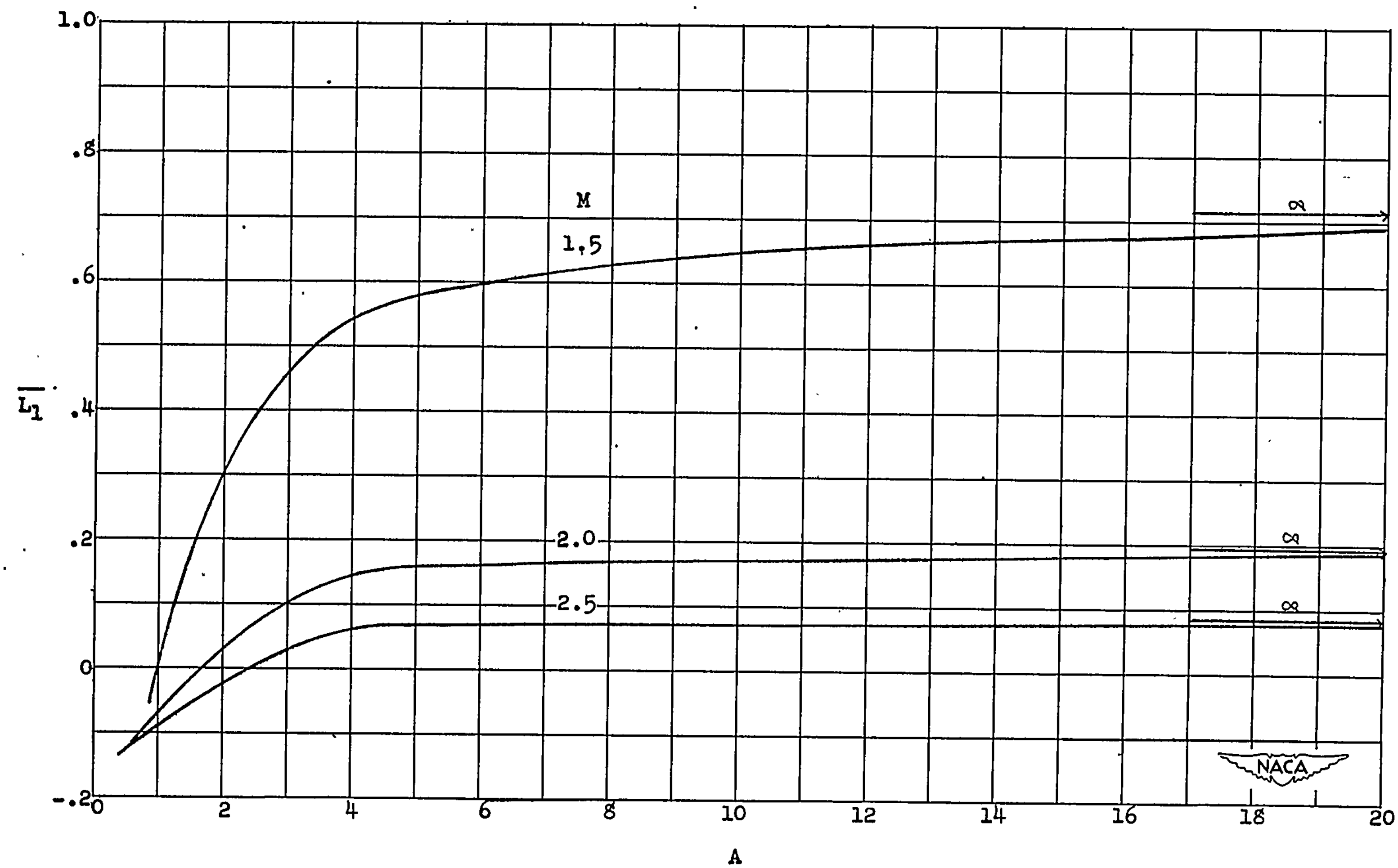
(c) $\overline{M_3}$.

Figure 5.- Continued.



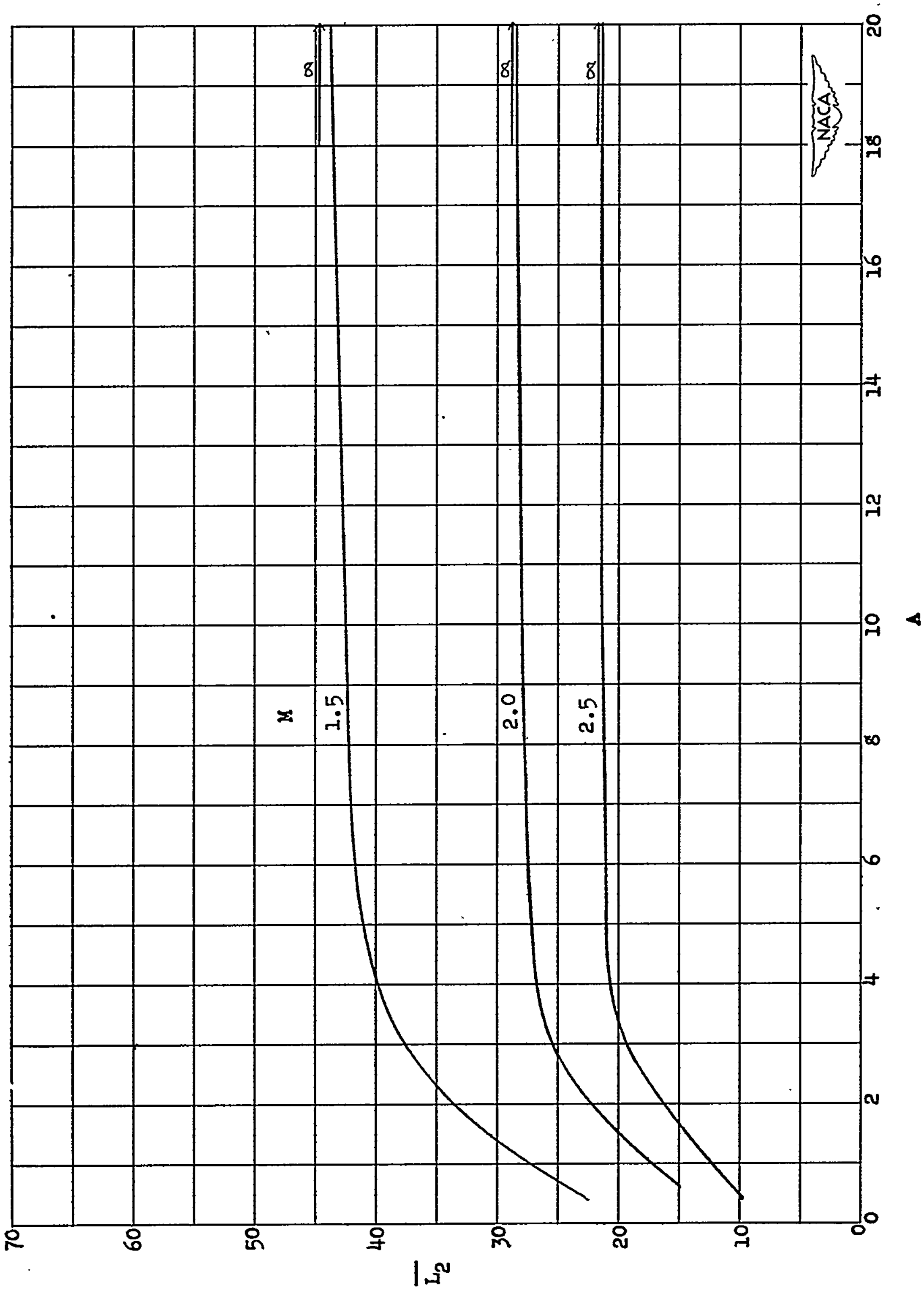
(d) $\overline{M_4}$.

Figure 5.- Concluded.



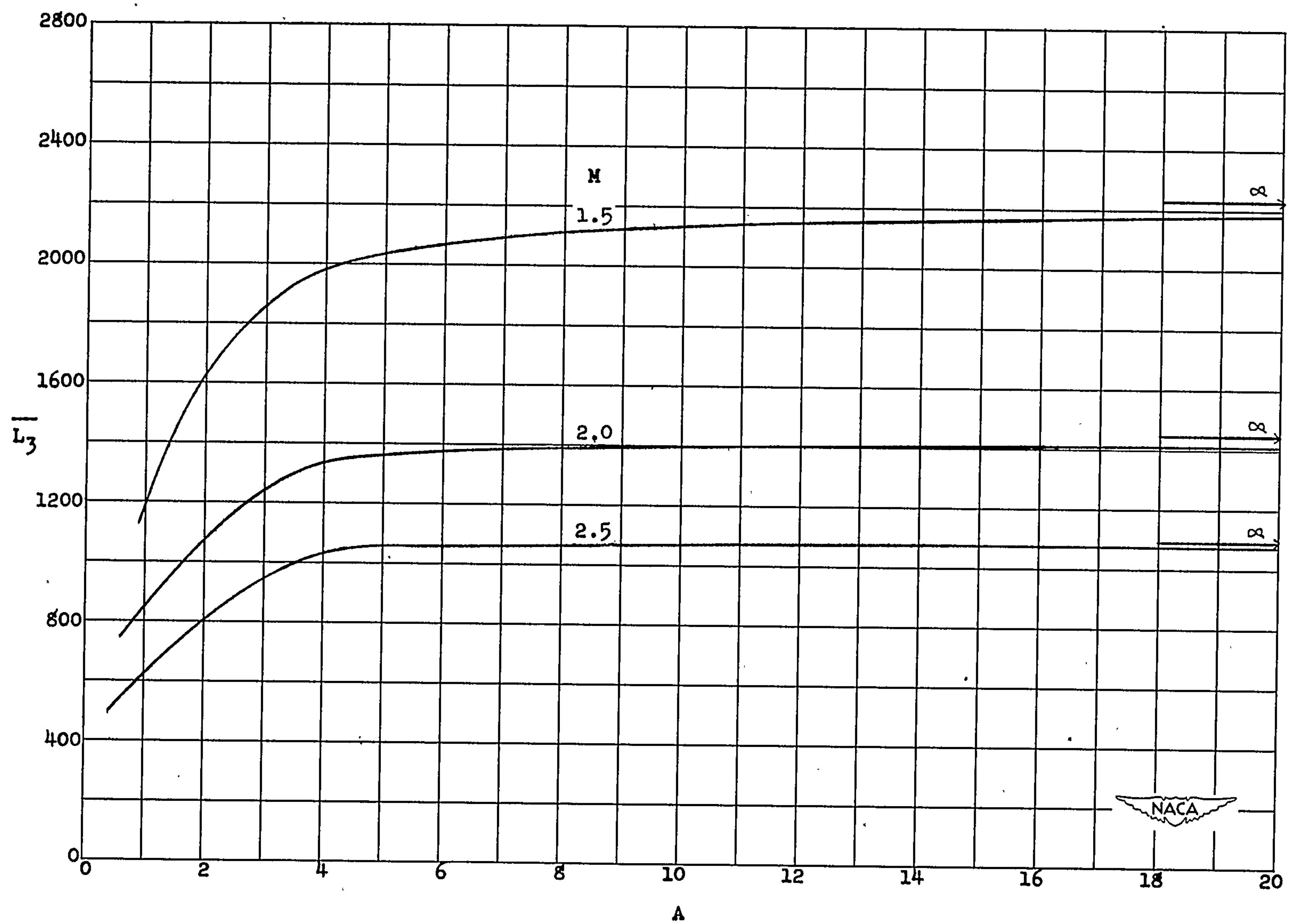
(a) $\overline{L_1}$.

Figure 6.- Components of total force coefficients as functions of A for $x_0 = 0.4$, $k = 0.02$, and various values of M.



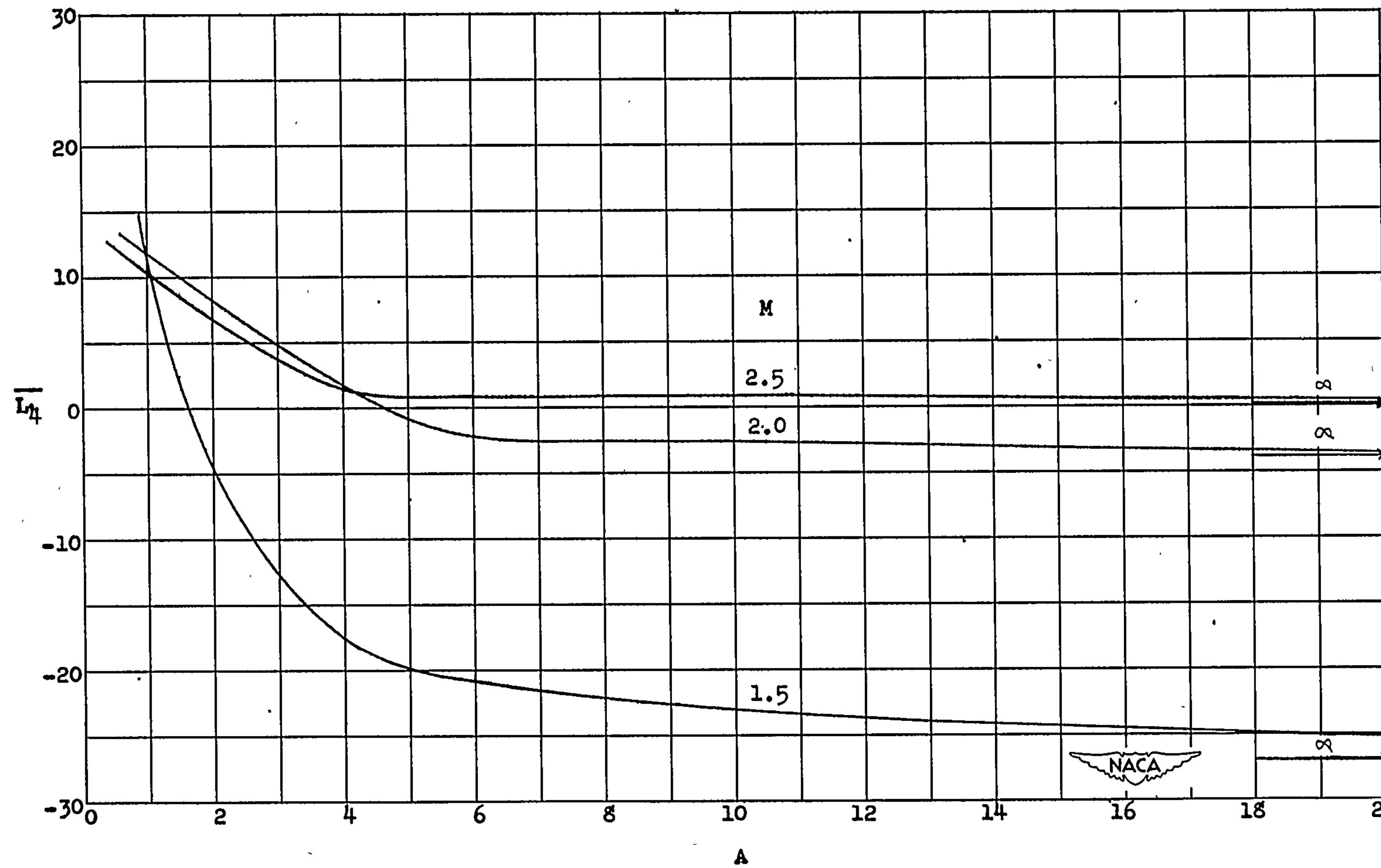
(b) $\overline{L_2}$.

Figure 6.- Continued.



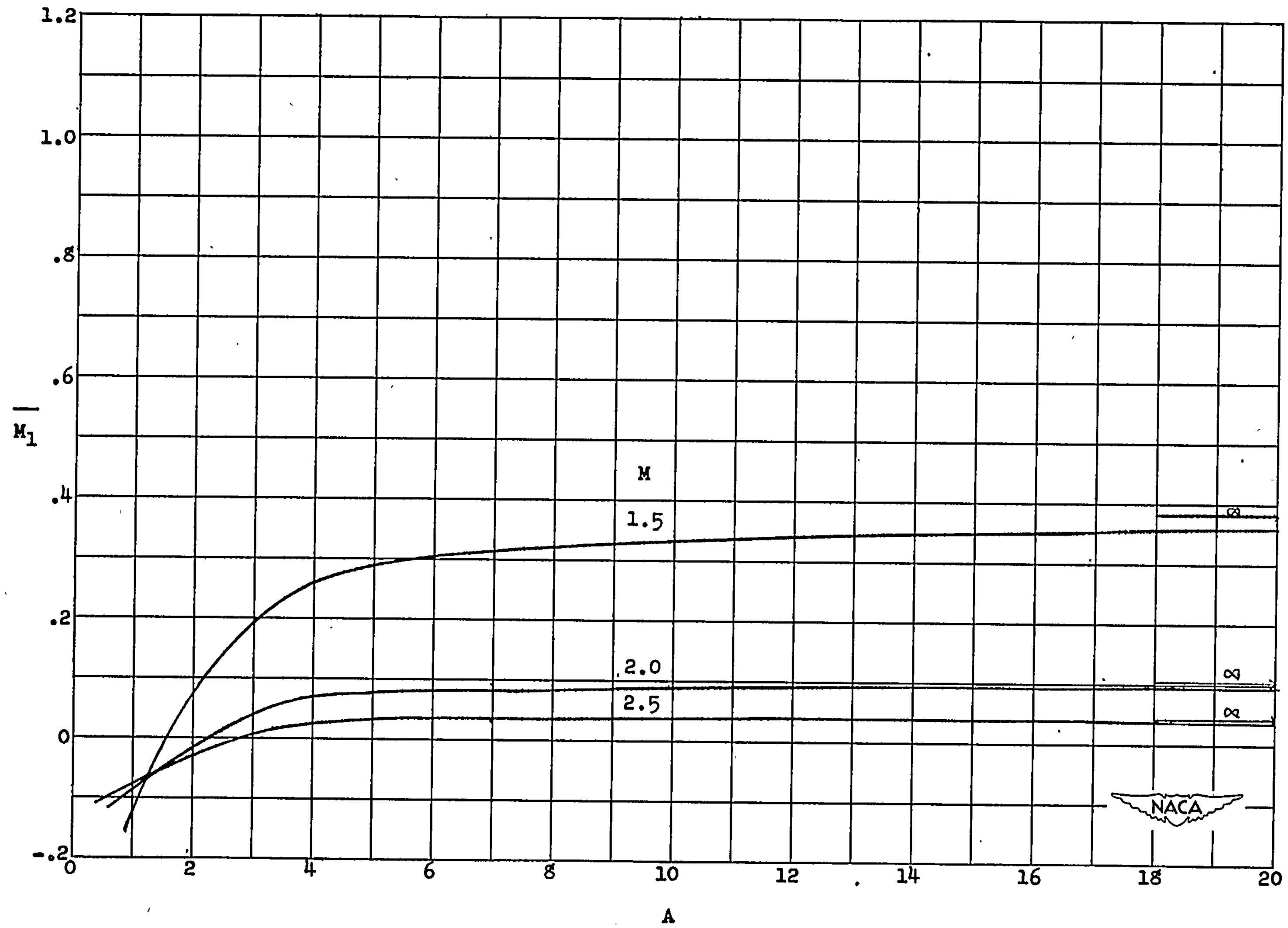
(c) $\overline{L_3}$.

Figure 6.- Continued.



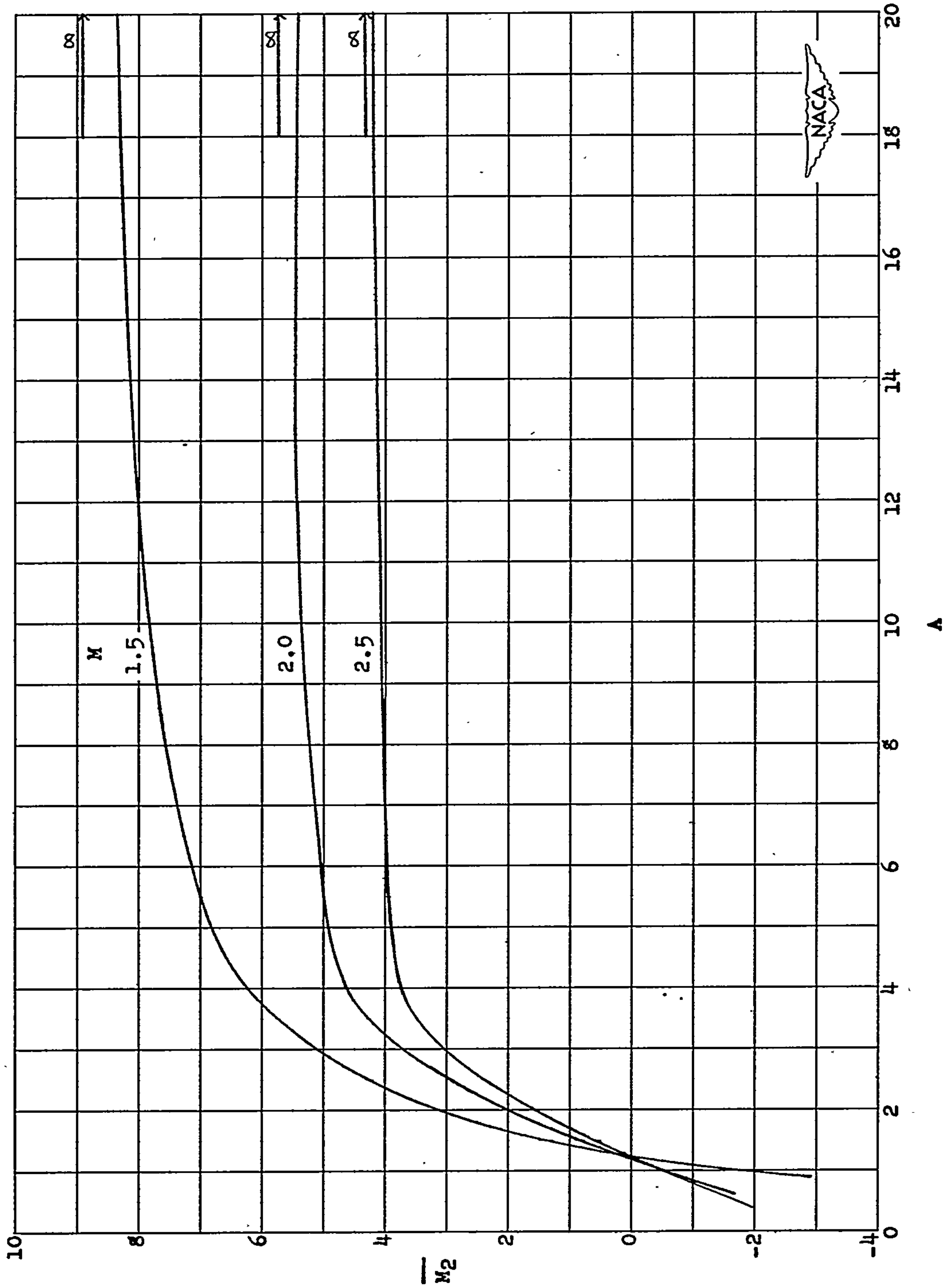
(d) $\overline{L_4}$.

Figure 6.- Concluded.



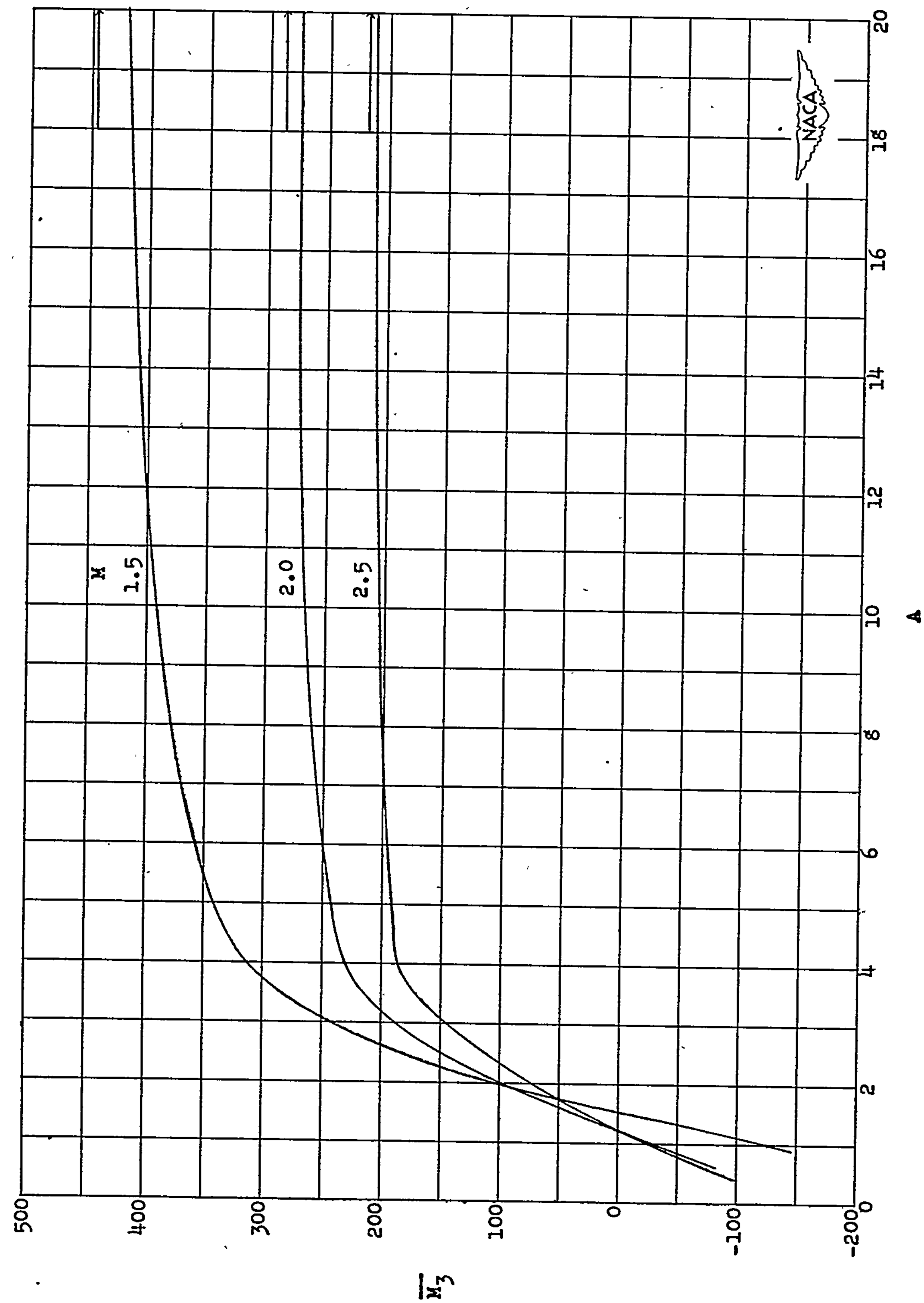
(a) $\overline{M_1}$.

Figure 7.- Components of total moment coefficients as functions of A for $x_0 = 0.4$, $k = 0.02$, and various values of M.



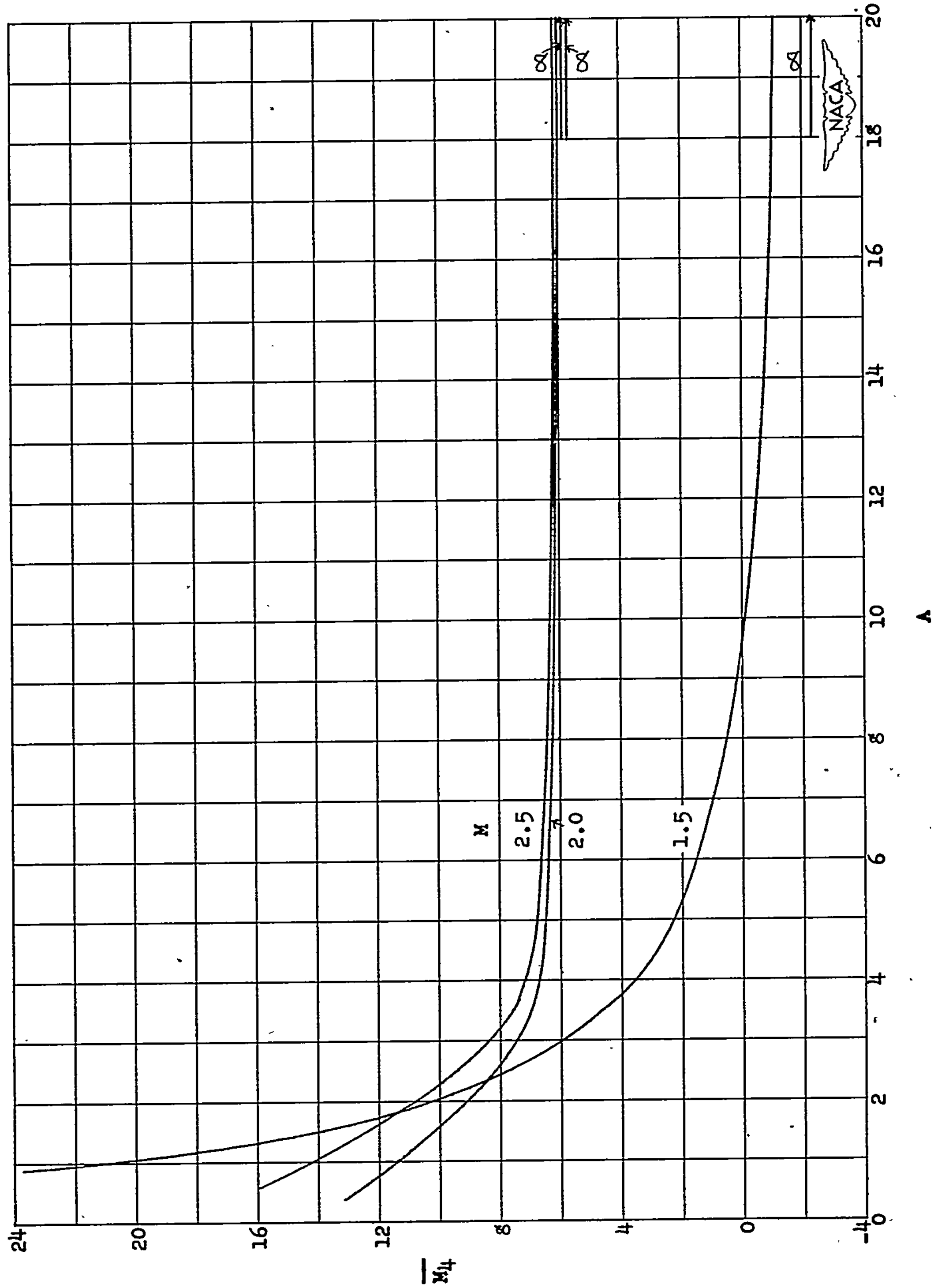
(b) $\overline{M_2}$.

Figure 7.- Continued.



(c) $\overline{M_3}$.

Figure 7.- Continued.



(d) $\overline{M_4}$.

Figure 7.- Concluded.

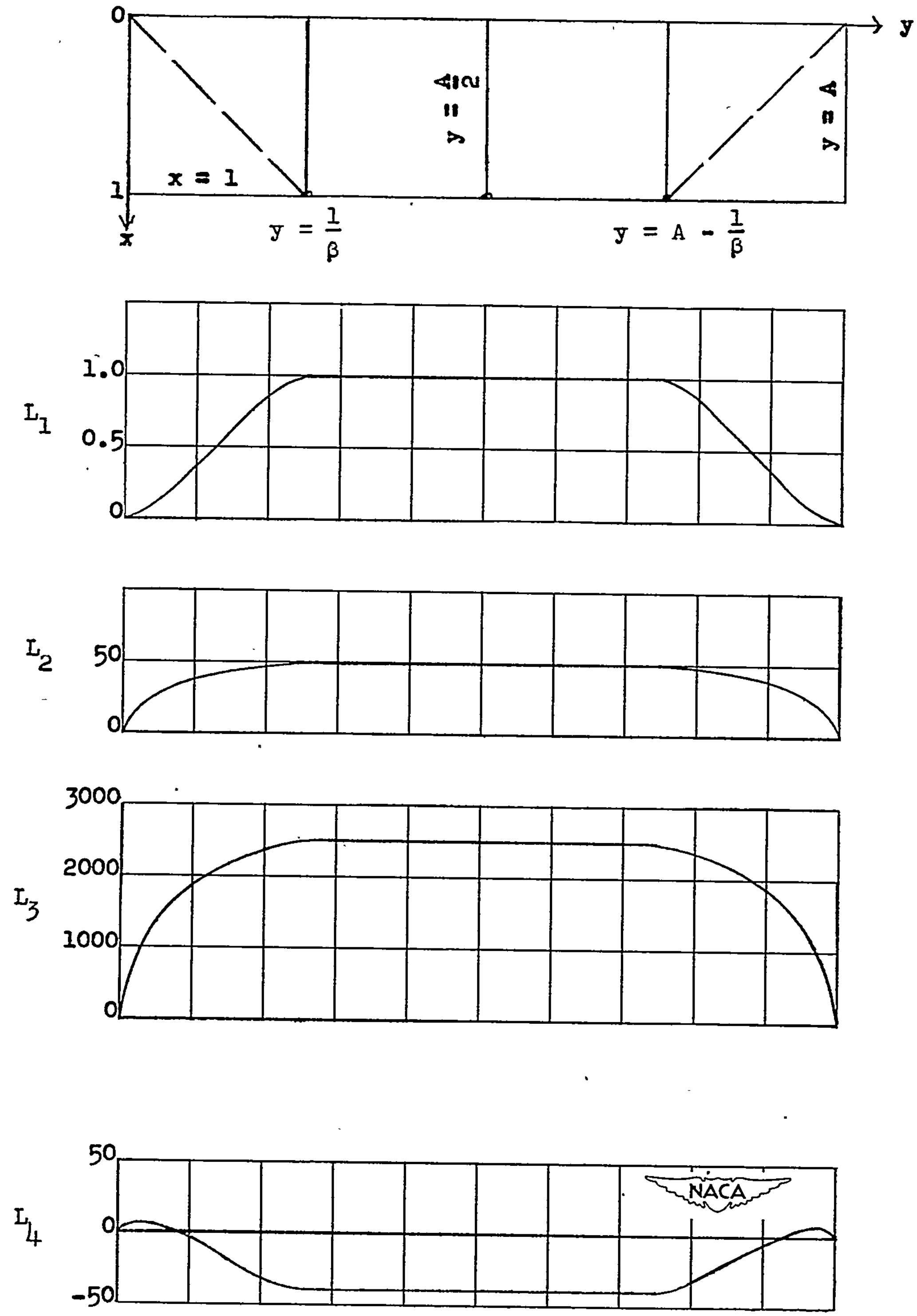


Figure 8.- Spanwise distribution of component of section force coefficients for $x_0 = 0.4$, $k = 0.02$, $M = 2$, and $A = 4$.

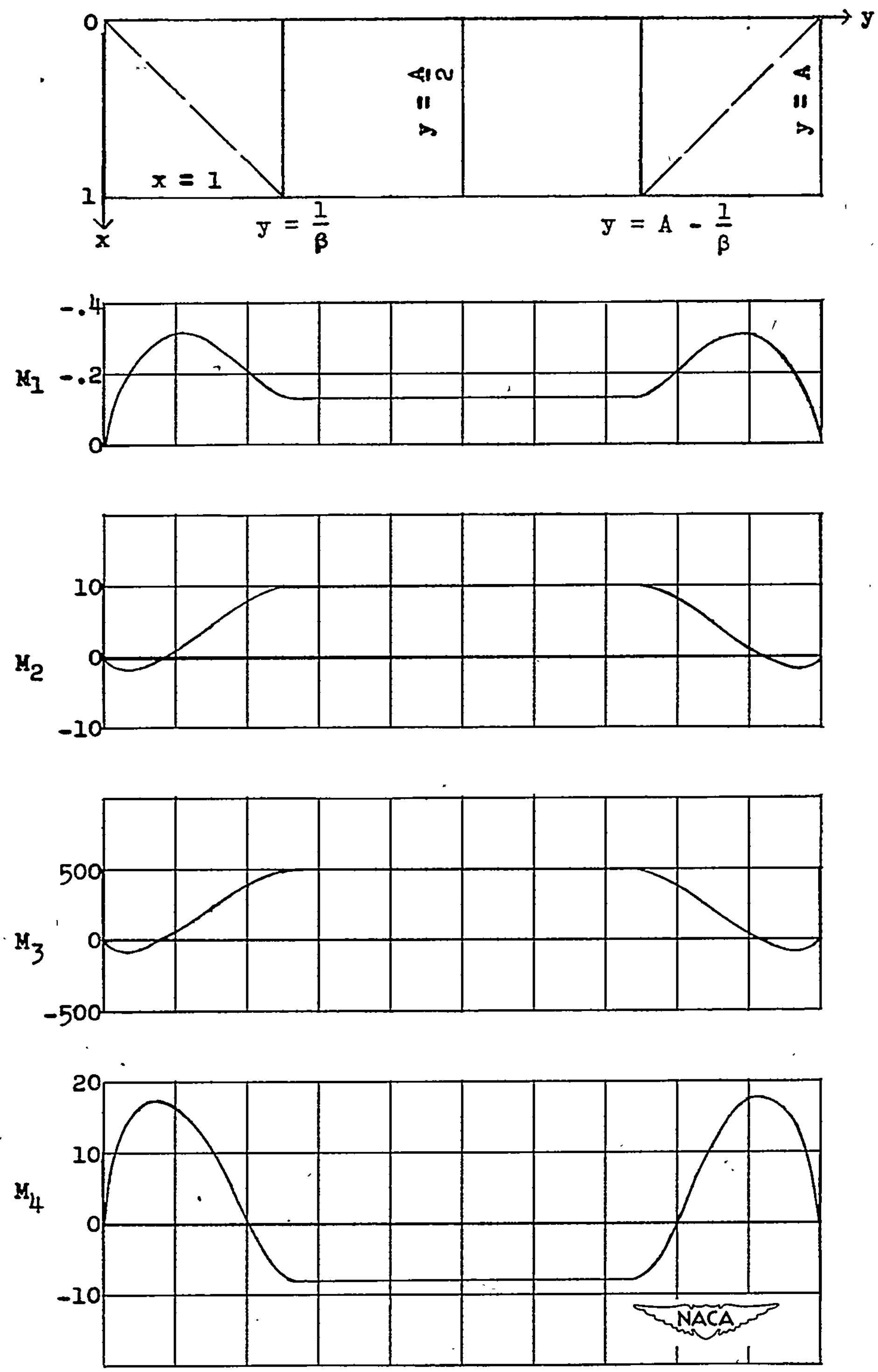


Figure 9.- Spanwise distribution of component of section moment coefficients for $x_0 = 0.4$, $k = 0.02$, $M = 2$, and $A = 4$.

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